Chapter 2. Series.

In this chapter we will learn the theory of "infinite sums".

Definition. Let $(a_n)_{n=0}^{\infty}$ be a sequence. Then the sequence $(S_n)_{n=0}^{\infty}$ defined by

$$S_n = \sum_{k=0}^{n} a_k, \quad n \geq 0,$$

is called a series, and S_n is called its n-th partial sum. The series $(S_n)_{n=0}^{\infty}$ is also denoted by the formal expression

$$\sum_{k=0}^{\infty} a_k.$$

If, however, the sequence $(S_n)_{n=0}^{\infty}$ has a limit, then we denote this limit by the same symbol

$$\sum_{k=0}^{\infty} a_k = \lim_{n \to \infty} \sum_{k=0}^{n} a_k.$$

In this case we call the series convergent, and $\sum_{k=0}^{\infty} a_k$ is called its sum.

If the series is not convergent, we call it divergent.

Remark Neither a series nor its partial sums have to start with index 0, e.g., we will regularly consider series of the form $\sum_{n=1}^{\infty} a_n$

The name of the index is not important either.

<u>Examples.</u> Let $a_n := x^n$, $n \geq 0$, where $x \in \mathbb{R}$ is a fixed number. Then

$\sum_{k=0}^{\infty} x^k$ is called the <u>geometric series</u>.

If $|x|<1$, then we have

$\lim_{n\to\infty} \sum_{k=0}^{n} x^k = \frac{1}{1-x}$, so the series is convergent with $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$.

If $x \in \mathbb{R} \setminus (-1,1)$, then the sequence of partial sums $s_n = \sum_{k=0}^{n} x^k$, $n \geq 0$,

is divergent.

ii) For the sequences $a_n := \frac{1}{n!}$, $n \geq 0$, and $b_n := \frac{1}{n^2}$, $n \geq 1$, we have already seen that

$$s_n = \sum_{k=0}^{n} a_k = \sum_{k=0}^{n} \frac{1}{k!}, \quad n \geq 0, \text{ and}$$

$$t_n = \sum_{k=1}^{n} b_k = \sum_{k=1}^{n} \frac{1}{k^2}, \quad n \geq 1, \text{ are}$$

bounded increasing sequences. Hence, by Thm 6, both series

$\sum_{k=0}^{\infty} \frac{1}{k!}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ are convergent.

<u>Theorem 12.</u> If the series $\sum_{k=0}^{\infty} a_k$ is convergent, then
$$\lim_{n\to\infty} a_n = 0.$$

<u>Proof.</u> Let us denote the sum of the series by s.

$$\forall n \geq 1: \quad s_n - s_{n-1} = a_n$$

$$\Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = s - s = 0. \qquad \square$$

<u>Theorem 13.</u> Let $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ be convergent series and $\alpha, \beta \in \mathbb{R}$. Then the series $\sum_{k=0}^{\infty} (\alpha a_k + \beta b_k)$ is convergent with sum
$$\sum_{k=0}^{\infty} (\alpha a_k + \beta b_k) = \alpha \sum_{k=0}^{\infty} a_k + \beta \sum_{k=0}^{\infty} b_k.$$

<u>Proof.</u> Using Thm 3 we obtain $\lim_{n\to\infty} \sum_{k=0}^{n} (\alpha a_k + \beta b_k)$
$$= \lim_{n\to\infty} \left\{ \alpha \sum_{k=0}^{n} a_k + \beta \sum_{k=0}^{n} b_k \right\}$$
$$= \alpha \lim_{n\to\infty} \sum_{k=0}^{n} a_k + \beta \lim_{n\to\infty} \sum_{k=0}^{n} b_k$$
$$= \alpha \sum_{k=0}^{\infty} a_k + \beta \sum_{k=0}^{\infty} b_k. \qquad \square$$

Theorem 14 (Cauchy's Condensation Test)

Let $(a_n)_{n=1}^{\infty}$ be a decreasing sequence with $a_n \geq 0$ for $n \in \mathbb{N}$. Then the series $\sum_{k=1}^{\infty} a_k$ is convergent if and only if the series $\sum_{k=0}^{\infty} 2^k a_{2^k}$ is convergent.

Proof. "$\Rightarrow$" Let us denote the sum of the series $\sum_{k=1}^{\infty} a_k$ by s. Then, for every $n \in \mathbb{N}$, we have

$$s = \sum_{k=1}^{\infty} a_k \geq \sum_{k=1}^{2^n} a_k = a_1 + a_2 + (a_3 + a_4) + (a_5 + \cdots + a_8) + \cdots + (a_{2^{n-1}+1} + \cdots + a_{2^n})$$

$$\geq \frac{a_1}{2} + a_2 + 2a_4 + 4a_8 + \cdots + 2^{n-1} a_{2^n}$$

$$\Rightarrow a_1 + 2a_2 + 4a_4 + \cdots + 2^n a_{2^n} \leq 2s.$$

It follows by Thm 6, that $\left(\sum_{k=1}^{n} 2^k a_{2^k}\right)_{n=1}^{\infty}$ is convergent.

"$\Leftarrow$" For $n, m \in \mathbb{N}$ with $n \leq 2^m$ we have

$$\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{2^m} a_k \leq a_1 + (a_2 + a_3) + (a_4 + \cdots + a_7) + \cdots + (a_{2^m} + \cdots + a_{2^{m+1}-1})$$

$$\leq a_1 + 2a_2 + 4a_4 + \cdots + 2^m a_{2^m} \leq \sum_{k=0}^{\infty} 2^k a_{2^k}.$$

Hence, by Thm 6, $\left(\sum_{k=1}^{n} a_k\right)_{n=1}^{\infty}$ is convergent. $\square$

Example. For any $\alpha \in \mathbb{R}$ we have that the series $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ is convergent if and only if $\alpha > 1$: If $\alpha \leq 0$, we have $\frac{1}{n^\alpha} \not\to 0$, so the series cannot be convergent according to Thm 12. If $\alpha > 0$, then according to Thm 14,

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \text{ is convergent} \iff \sum_{n=0}^{\infty} 2^n \frac{1}{2^{\alpha n}} \text{ is convergent.}$$

We have $\sum_{n=0}^{\infty} 2^n \frac{1}{2^{\alpha n}} = \sum_{n=0}^{\infty} \left(\frac{1}{2^{\alpha-1}}\right)^n$, which is convergent if and only if $\frac{1}{2^{\alpha-1}} < 1$ $\iff \alpha > 1$.

In particular, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. This series is called the <u>harmonic series</u>.

Moreover, the function $\zeta : (1, \infty) \to \mathbb{R}$, defined by $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$, is called <u>Riemann zeta function</u>.

Theorem 15 (Dirichlet's Test)

Let $(a_n)_{n=0}^{\infty}$ be a sequence such that the sequence $s_n = \sum_{k=0}^{n} a_k$, $n \geq 0$, is bounded, and let $(b_n)_{n=0}^{\infty}$ be a decreasing sequence converging to 0.
Then the series $\sum_{k=0}^{\infty} a_k b_k$ is convergent.

Proof. We know s_n is bounded

$$\Rightarrow \exists M > 0 \; \forall n \geq 0 : \; \left| \sum_{k=0}^{n} a_k \right| \leq M.$$

We use Abel's summation formula: For $n \geq m \geq 0$

$$\sum_{k=m+1}^{n} a_k b_k = b_{n+1} \sum_{k=m+1}^{n} a_k + \sum_{k=m+1}^{n} \left(\sum_{\nu=m+1}^{k} a_\nu \right)(b_k - b_{k+1}) \quad (*)$$

$$\sum_{k=m+1}^{n} \left(\sum_{\nu=m+1}^{k} a_\nu \right)(b_k - b_{k+1}) = \sum_{k=m+1}^{n} \left(\sum_{\nu=m+1}^{k} a_\nu \right) b_k - \sum_{k=m+1}^{n} \left(\sum_{\nu=m+1}^{k} a_\nu \right) b_{k+1}$$

$$= a_{m+1} b_{m+1} + \sum_{k=m+2}^{n} \left(\sum_{\nu=m+1}^{k} a_\nu \right) b_k - \sum_{k=m+2}^{n+1} \left(\sum_{\nu=m+1}^{k-1} a_\nu \right) b_k$$

$$= a_{m+1} b_{m+1} + \sum_{k=m+2}^{n} a_k b_k - \left(\sum_{\nu=m+1}^{n} a_\nu \right) b_{n+1}$$

$$= \sum_{k=m+1}^{n} a_k b_k - b_{n+1} \sum_{k=m+1}^{n} a_k .$$

We now show that the sequence $t_n = \sum_{k=0}^{n} a_k b_k$, $n \geq 0$, is a Cauchy sequence.

Let $\varepsilon > 0$. We know $b_n \to 0$, so

$$\exists N \in \mathbb{N}\ \forall n \geq N: \quad b_n < \frac{\varepsilon}{2M}.$$

For $m \geq n \geq N$ we have

$$|t_n - t_m| = \left| \sum_{k=n+1}^{m} a_k b_k \right|$$

$$\underset{(*)}{\leq} b_{m+1} \underbrace{\left| \sum_{k=n+1}^{m} a_k \right|}_{= s_m - s_n} + \sum_{k=n+1}^{m} \underbrace{\left| \sum_{\nu=n+1}^{k} a_\nu \right|}_{= s_k - s_n} (b_k - b_{k+1})$$

$$\leq 2M \left(b_{m+1} + \sum_{k=n+1}^{m} (b_k - b_{k+1}) \right) = 2M\, b_{n+1} < \varepsilon. \qquad \square$$

<u>Theorem 16.</u> (Leibniz's Test)

Let $(b_n)_{n=0}^{\infty}$ be a decreasing sequence converging to 0. Then the series

$$\sum_{n=0}^{\infty} (-1)^n b_n \quad \text{is convergent.}$$

<u>Proof.</u> Set $a_n := (-1)^n$, $n \geq 0$, and use Thm 15. $\square$

<u>Example.</u> The <u>alternating harmonic series</u>

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \quad \text{is convergent by Thm 16.}$$

<u>Definition.</u> Let $(a_n)_{n=0}^{\infty}$ be a sequence. We say that the series $\sum_{n=0}^{\infty} a_n$ is <u>absolutely convergent</u> if $\sum_{n=0}^{\infty} |a_n|$ is convergent.

<u>Remarks.</u> i) The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is convergent but not absolutely convergent.

ii) In the case $a_n \geq 0$, $n \geq 0$, the notions of convergence and absolute convergence coincide.

<u>Theorem 17.</u> Every absolutely convergent series is convergent.

<u>Proof.</u> Let $\sum_{n=0}^{\infty} a_n$ be absolutely convergent.

We want to show that $s_n = \sum_{k=0}^{n} a_k$, $n \geq 0$, is convergent. Let $\varepsilon > 0$. We know by Thm 8 that $\sum_{k=0}^{n} |a_k|$, $n \geq 0$, is a Cauchy sequence, hence

$$\exists N \in \mathbb{N} \ \forall n \geq m \geq N: \quad \sum_{k=m+1}^{n} |a_k|$$

$$= \left| \sum_{k=0}^{n} |a_k| - \sum_{k=0}^{m} |a_k| \right| < \varepsilon.$$

$$\Rightarrow \forall n \geq m \geq N: \quad |s_n - s_m| = \left| \sum_{k=m+1}^{n} a_k \right| \leq \sum_{k=m+1}^{n} |a_k| < \varepsilon.$$

$\Rightarrow (s_n)_{n=0}^{\infty}$ is Cauchy $\Rightarrow$ (Thm 8) (s_n) convergent. $\square$

<u>Theorem 18.</u> If $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent and $\alpha, \beta \in \mathbb{R}$, then the series $\sum_{n=0}^{\infty} (\alpha a_n + \beta b_n)$ is absolutely convergent.

<u>Proof.</u> Follows immediately using the triangle inequality. $\square$

<u>Theorem 19.</u> (Comparison Test)

Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be sequences and $N \in \mathbb{N}$.

i) If $\sum_{n=0}^{\infty} b_n$ is absolutely convergent and if there exists a constant $K > 0$ such that

$|a_n| \leq K |b_n|$ for all $n \geq N$, then

$\sum_{n=0}^{\infty} a_n$ is absolutely convergent.

ii) If $\sum_{n=0}^{\infty} |a_n|$ is divergent and if there is a constant $K > 0$ such that $|b_n| \geq K |a_n|$ for all $n \geq N$, then $\sum_{n=0}^{\infty} |b_n|$ is divergent.

iii) If $a_n, b_n \geq 0$, $n \geq 0$, and $\lim_{n \to \infty} \frac{a_n}{b_n} = c$ with $c \in (0, \infty)$, then

$\sum_{n=0}^{\infty} a_n$ is convergent $\Leftrightarrow$ $\sum_{n=0}^{\infty} b_n$ is convergent.

Proof. i) For $m \geq N$ we have

$$\sum_{k=0}^{m} |a_k| = \sum_{k=0}^{N-1} |a_k| + \sum_{k=N}^{m} |a_k| \leq \sum_{k=0}^{N-1} |a_k| + K \sum_{k=N}^{m} |b_k|$$

$$\leq \sum_{k=0}^{N-1} |a_k| + K \sum_{k=N}^{\infty} |b_k|,$$

so the sequence $\sum_{k=0}^{m} |a_k|$, $m \geq 0$, is increasing and bounded

$\Rightarrow$ (Thm 6) $\sum_{n=0}^{\infty} |a_n|$ is convergent.

ii) For $m \geq N$ we have

$$\sum_{k=0}^{m} |b_k| = \sum_{k=0}^{N-1} |b_k| + \sum_{k=N}^{m} |b_k| \geq \sum_{k=0}^{N-1} |b_k| + K \underbrace{\sum_{k=N}^{m} |a_k|}_{\to \infty,\ m \to \infty},$$

$$\Rightarrow \sum_{k=0}^{m} |b_k| \to \infty, \quad m \to \infty.$$

iii) We know $\lim_{n\to\infty} \frac{a_n}{b_n} = c \in (0, \infty)$

$\Rightarrow \exists N_1 \in \mathbb{N}\ \forall n \geq N_1 : \frac{1}{2} c \leq \frac{a_n}{b_n} \leq \frac{3}{2} c$

$\Rightarrow \forall n \geq N_1 : \frac{1}{2} c\, b_n \leq a_n \leq \frac{3}{2} c\, b_n$

Now the statement follows from i) and ii). $\square$

Examples. i) The series $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ is convergent by part i) of Thm 19:

$$\forall n \geq 1 : \frac{\sqrt{n}}{n^2+1} \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}},$$