For $n \geq 2$ we have

$$0 < \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \sum_{k=2}^{n} \frac{1}{k^2} \leq 1 + \sum_{k=2}^{n} \frac{1}{k(k-1)}$$

$$= 1 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k} \right) = 1 + 1 - \frac{1}{n} = 2 - \frac{1}{n} < 2.$$

Hence, the sequence $a_n := \sum_{k=1}^{n} \frac{1}{k^2}$, $n \geq 1$, is bounded and strictly increasing, so by Thm 6 we know

$$\lim_{n \to \infty} a_n \leq 2.$$

ii) Euler's number e: Let (e_n) be defined by

$$e_n := \left(1 + \frac{1}{n}\right)^n, \quad n \geq 1.$$

α) In order to show that (e_n) is increasing, we first recall Bernoulli's inequality

$\forall n \geq 2$ and $x \geq -1$: $(1+x)^n \geq 1 + nx$

with equality if and only if $x = 0$.

Hence, for $x = -\frac{1}{n^2}$, $n \geq 2$, we obtain

$$\left(1 - \frac{1}{n^2}\right)^n > 1 - \frac{1}{n}.$$

$$\Rightarrow \left(1 - \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right)^n > 1 - \frac{1}{n}, \quad n \geq 2$$

$$\Rightarrow e_n = \left(1 + \frac{1}{n}\right)^n > \left(1 - \frac{1}{n}\right)^{1-n} = \left(\frac{n-1}{n}\right)^{1-n} = \left(\frac{n}{n-1}\right)^{n-1}$$

$$= e_{n-1}, \quad n \geq 2.$$

Hence, (e_n) is strictly increasing.

β) Next, we show that (e_n) is bounded. For $n \geq 2$ we have

$$0 < e_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k}$$

$$= 1 + 1 + \sum_{k=2}^{n} \frac{1}{k!} \underbrace{\frac{n(n-1)\cdots(n-k+1)}{n \; n \; \cdots \; n}}_{\leq 1} \leq 2 + \sum_{k=2}^{n} \frac{1}{k!}$$

$$\leq 2 + \sum_{k=2}^{n} \frac{1}{k(k-1)} = 2 + \sum_{k=2}^{n} \left(\frac{1}{k-1} - \frac{1}{k}\right)$$

$$= 2 + 1 - \frac{1}{n} < 3.$$

Thus, by Thm 6, it follows that $\lim\limits_{n\to\infty} e_n$ exists.

<u>Definition</u>. <u>Euler's number</u> e is defined by

$$e := \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n.$$

<u>Theorem 7</u> (Bolzano-Weierstraß)

Every bounded sequence has a convergent subsequence.

<u>Proof</u>. The proof follows immediately from Thm 5 in combination with Thm 6. $\square$

Next, we extend the notion of limits to the "infinite" case.

<u>Definition.</u> Let (a_n) be a sequence of real numbers.

i) We write $\lim_{n\to\infty} a_n = \infty$, or $a_n \to \infty$, if

$$\forall M > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N : \ a_n \geq M.$$

In this case we say that (a_n) <u>tends to infinity, as n tends to infinity</u>.

ii) We write $\lim_{n\to\infty} a_n = -\infty$, or $a_n \to -\infty$, if

$$\forall M < 0 \ \exists N \in \mathbb{N} \ \forall n \geq N : \ a_n \leq M.$$

In this case we say that (a_n) <u>tends to negative infinity, as n tends to infinity</u>.

<u>Examples.</u> i) The sequence $a_n := n$, $n \geq 1$, is divergent, but it tends to infinity, as n tends to infinity

$$\lim_{n\to\infty} n = \infty.$$

We use this term, although the limit does not exist, to indicate that the sequence grows beyond every bound and "approaches" infinity.

ii) The sequence $b_n := (-1)^n$, $n \geq 1$, is divergent, and it tends neither to infinity nor to negative infinity.

<u>Definition.</u> A sequence (a_n) is called a <u>Cauchy sequence</u> if we have

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n, m \geq N : |a_n - a_m| < \varepsilon.$$

<u>Theorem 8.</u> A sequence of real numbers is convergent if and only if it is a Cauchy sequence.

<u>Proof.</u> Let $(a_n) = (a_n)_{n=1}^{\infty}$ be a sequence.

"$\Rightarrow$" If (a_n) has the limit $a \in \mathbb{R}$, then

$$\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N : |a_n - a| < \frac{\varepsilon}{2}.$$

$$\Rightarrow \forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n, m \geq N :$$

$$|a_n - a_m| \leq |a_n - a| + |a_m - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

"$\Leftarrow$" Let (a_n) be a Cauchy sequence.

α) We show that (a_n) is bounded. We know

$$\exists N \in \mathbb{N} \; \forall n \geq N : |a_n - a_N| < 1$$

$$\Rightarrow \forall n \geq N : |a_n| = |a_n - a_N + a_N| \leq |a_n - a_N| + |a_N|$$

$$\leq 1 + \max\{|a_1|, \dots, |a_N|\}.$$

Hence, for all $n \geq 1$:

$$|a_n| \leq 1 + \max\{|a_1|, \dots, |a_N|\}.$$

β) Using Thm 7, we obtain a convergent subsequence $(a_{n_k})_{k=1}^{\infty}$ with limit $a \in \mathbb{R}$. Let $\varepsilon > 0$.

$\Rightarrow \exists N \in \mathbb{N}\ \forall n, m \geq N: \ |a_n - a_m| < \frac{\varepsilon}{2}$.

Moreover, we know

$\exists K \in \mathbb{N}: \ n_K \geq N$ and $|a_{n_K} - a| < \frac{\varepsilon}{2}$.

$\Rightarrow \forall n \geq N: \ |a_n - a| \leq |a_n - a_{n_K}| + |a_{n_K} - a| < \varepsilon$.

$\Rightarrow \lim\limits_{n \to \infty} a_n = a$. □

Example. Let (a_n) be a sequence such that

$$|a_{n+1} - a_n| \leq \frac{1}{2^n}, \quad n \geq 1.$$

In order to prove that (a_n) is convergent, it is sufficient to prove that (a_n) is a Cauchy sequence.

Let $\varepsilon > 0$. $\Rightarrow \exists N \in \mathbb{N}\ \forall m \geq N: \ \frac{1}{2^{m-1}} < \varepsilon$.

$$\Rightarrow \forall n \geq m \geq N: \ |a_n - a_m| = \left| \sum_{k=m}^{n-1} (a_{k+1} - a_k) \right|$$

$$\leq \sum_{k=m}^{n-1} |a_{k+1} - a_k| \leq \sum_{k=m}^{n-1} \left(\frac{1}{2}\right)^k = \frac{\left(\frac{1}{2}\right)^m - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}}$$

$$= 2\left(\left(\frac{1}{2}\right)^m - \left(\frac{1}{2}\right)^n\right) < \frac{1}{2^{m-1}} < \varepsilon.$$

$\Rightarrow$ (Thm 8) (a_n) is convergent.

Next, we turn to locations where the terms of a sequence cluster.

<u>Definition</u>. Let (a_n) be a sequence. A point $a \in \mathbb{R}$ is called <u>cluster point</u> of (a_n) if

$$\forall \varepsilon > 0 \ \forall N \in \mathbb{N} \ \exists n \geq N: \ a_n \in U_\varepsilon(a).$$

<u>Remark.</u> A cluster point of (a_n) is a point $a \in \mathbb{R}$ such that, for every $\varepsilon > 0$, we have

$a_n \in U_\varepsilon(a)$ for infinitely many indices n.

<u>Theorem 9.</u> Let (a_n) be a sequence of real numbers, and let $a \in \mathbb{R}$. Then the point a is a cluster point if and only if there exists a subsequence (a_{n_k}) with $\lim\limits_{k \to \infty} a_{n_k} = a$.

<u>Proof.</u> "$\Rightarrow$" Let a be a cluster point of (a_n), then

$\exists n_1 \geq 1: \quad a_{n_1} \in U_1(a)$

$\Rightarrow \exists n_2 \geq n_1 + 1: \quad a_{n_2} \in U_{\frac{1}{2}}(a)$

$\vdots$

$\Rightarrow \exists n_k \geq n_{k-1} + 1: \quad a_{n_k} \in U_{\frac{1}{k}}(a)$

Inductively, we construct a strictly increasing sequence (n_k) of natural numbers such that

for all $k \geq 1$: $\quad a_{n_k} \in U_{\frac{1}{k}}(a)$

$\Rightarrow (a_{n_k})$ is a convergent subsequence with $a_{n_k} \to a$.

"$\Leftarrow$" Let (a_{n_k}) be a convergent subsequence with limit $a \in \mathbb{R}$. Let $\varepsilon > 0$, and $N \in \mathbb{N}$.

We want to show that $\exists n \geq N: a_n \in U_\varepsilon(a)$.

We know $\exists K \in \mathbb{N}\ \forall k \geq K: \quad a_{n_k} \in U_\varepsilon(a)$

We choose any $k_0 \geq K$ such that $n_{k_0} \geq N$, then $a_{n_{k_0}} \in U_\varepsilon(a)$. □

<u>Example</u>. We consider the divergent sequence (a_n) with $a_n = (-1)^n + \frac{1}{n}$, $n \geq 1$. We can identify its cluster points by choosing convergent subsequences:

$n_k = 2k \Rightarrow a_{n_k} = 1 + \frac{1}{2k} \to 1$, so 1 is a cluster point,

$n_k = 2k+1 \Rightarrow a_{n_k} = -1 + \frac{1}{2k+1} \to -1$, so -1 is a cluster point, and it is not difficult to show that there are no other cluster points.

<u>Theorem 10</u>. Every bounded sequence (a_n) has a greatest and a smallest cluster point.

<u>Proof</u>. We denote the set of cluster points of (a_n) by C. W.l.o.g. we only show that $\max C$ exists.

We first observe that $C \neq \emptyset$, which follows from Thm 7 in combination with Thm 9.

i) (a_n) is bounded $\Rightarrow \exists M > 0 \ \forall n \geq 1: |a_n| \leq M$.
Let $x \in C$, then by Thm 9, there exists a subsequence (a_{n_k}) with $a_{n_k} \to x$.
$\Rightarrow$ (Thm 4) $|x| = \lim_{k \to \infty} |a_{n_k}| \leq M$
$\Rightarrow \sup C \in \mathbb{R}$.

ii) Let us denote $\alpha := \sup C$. We want to show $\alpha \in C$.
Let $\varepsilon > 0$, then we find a $\beta \in C$ such that

$$\alpha - \frac{\varepsilon}{2} < \beta \leq \alpha.$$

$\Rightarrow \exists$ infinitely many n: $a_n \in U_{\frac{\varepsilon}{2}}(\beta) \subset U_\varepsilon(\alpha)$
$\Rightarrow \alpha \in C$. $\square$

<u>Remark</u>. The proof of Thm 10 shows that the greatest cluster point exists if the sequence is bounded above and has at least one cluster point. Likewise, the smallest cluster point exists if the sequence is bounded below and has at least one cluster point.

<u>Definition</u>. Let (a_n) be a sequence, and let C be the set of cluster points of (a_n).

We call

$$\limsup_{n\to\infty} a_n := \begin{cases} \max C, & \text{if } C \neq \emptyset \text{ and } (a_n) \text{ bounded above} \\ \infty, & \text{if } (a_n) \text{ is not bounded above} \\ -\infty, & \text{if } a_n \to -\infty \end{cases}$$

the <u>limit superior</u> of (a_n), and

$$\liminf_{n\to\infty} a_n := \begin{cases} \min C, & \text{if } C \neq \emptyset \text{ and } (a_n) \text{ bounded below} \\ -\infty, & \text{if } (a_n) \text{ is not bounded below} \\ \infty, & \text{if } a_n \to \infty \end{cases}$$

the <u>limit inferior</u> of (a_n).

<u>Remark</u> For every sequence (a_n) we have

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n.$$

<u>Theorem 11.</u> For every sequence (a_n) we have:

(a_n) is convergent $\iff$ (a_n) is bounded and $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$,

and in this case $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n$.

<u>Proof.</u> "$\Rightarrow$" By Thm 3, (a_n) is bounded, and for every subsequence (a_{n_k}) we have

$$a_{n_k} \to \lim_{n\to\infty} a_n \Rightarrow C = \{\lim_{n\to\infty} a_n\}$$

$$\Rightarrow \max C = \min C = \lim_{n\to\infty} a_n.$$

"$\Leftarrow$" We set $a := \limsup_{n\to\infty} a_n \in \mathbb{R}$, and let $\varepsilon > 0$.
The inequality $a_n \geq a + \varepsilon$ can only hold true for a finite number of indices (otherwise we could find a convergent subsequence with limit $\geq a + \varepsilon$, which would imply a cluster point greater than $\limsup_{n\to\infty} a_n$).

$\Rightarrow \exists N_1 \in \mathbb{N}\ \forall n \geq N_1 : \quad a_n < a + \varepsilon.$

Using the same reasoning, we find

$\exists N_2 \in \mathbb{N}\ \forall n \geq N_2 : \quad a_n > a - \varepsilon.$

$\Rightarrow \forall n \geq \max\{N_1, N_2\} : \quad a - \varepsilon < a_n < a + \varepsilon.$

$\Rightarrow \lim_{n\to\infty} a_n = a.$ $\square$

<u>Example.</u> Let us consider the sequence $(a_n)_{n=1}^{\infty}$ with $a_n :=$ remainder when n is divided by 4, i.e. $(a_n) = (1, 2, 3, 0, 1, 2, 3, 0, \dots)$.

For $l \in \{0, 1, 2, 3\}$ we have

$(a_{4k+l})_{k=1}^{\infty} = (l, l, \dots) \to l$, so we know

that the set of cluster points is $C = \{0, 1, 2, 3\}$.

It follows that

$$\limsup_{n\to\infty} a_n = 3 \quad \text{and} \quad \liminf_{n\to\infty} a_n = 0.$$