Theorem 4. Let (a_n), (b_n) and (c_n) be sequences.

i) If $a_n \to a$, $b_n \to b$, and for all indices n: $a_n \leq b_n$, then we have $a \leq b$.

ii) (Squeeze Theorem)
If $a_n \to a$, $b_n \to a$, and for all indices n:
$$a_n \leq c_n \leq b_n, \text{ then } c_n \to a.$$

iii) If for all indices n: $|a_n - a| \leq b_n$, and $b_n \to 0$, then $a_n \to a$.

iv) If (b_n) is bounded and $a_n \to 0$, then $a_n b_n \to 0$.

Proof. W.l.o.g. we assume that all sequences start at $n=1$.

i) We prove this part by contradiction.
Assume: $a > b$. We set $\varepsilon := \frac{a-b}{2} > 0$.
$a_n \to a \Rightarrow \exists N_1 \in \mathbb{N}\ \forall n \geq N_1 : a_n \in \mathcal{U}_\varepsilon(a)$
$b_n \to b \Rightarrow \exists N_2 \in \mathbb{N}\ \forall n \geq N_2 : b_n \in \mathcal{U}_\varepsilon(b)$

$\Rightarrow \forall n \geq \max\{N_1, N_2\}$:

$$a_n > a - \varepsilon = \frac{a+b}{2} = b + \varepsilon > b_n,$$ which is a contrad.

ii) $a_n \to a$ and $b_n \to a \Rightarrow \exists N \in \mathbb{N}\ \forall n \geq N$:

$$|a_n - a| < \frac{\varepsilon}{2} \text{ and } |b_n - a| < \frac{\varepsilon}{2}$$

for any given $\varepsilon > 0$.

$\Rightarrow \forall n \geq N: \quad a_n - a \leq c_n - a \leq b_n - a$

$\Rightarrow \forall n \geq N: \quad |c_n - a| \leq \max\{|a_n - a|, |b_n - a|\}$

$\leq |a_n - a| + |b_n - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$

iii) follows directly from the definition of convergence.

iv) Let $\varepsilon > 0$. We know (b_n) is bounded

$\Rightarrow \exists M > 0 \quad \forall n \geq 1: \quad |b_n| \leq M.$

As (a_n) converges to 0, we know

$\exists N \in \mathbb{N} \; \forall n \geq N: \; |a_n| < \frac{\varepsilon}{M}.$

$\Rightarrow \forall n \geq N: \; |a_n b_n| \leq M |a_n| < M \frac{\varepsilon}{M} = \varepsilon.$ $\square$

Examples. i) We have $\frac{n-1}{n+1} = \frac{1 - 1/n}{1 + 1/n} \to \frac{1}{1} = 1,$

where we use Thm 3, iv).

ii) Let $\alpha > 0$ be a fixed positive number and

$a_n := \frac{1}{n^\alpha}, \; n \in \mathbb{N}$. We show $a_n \to 0$.

Let $\varepsilon > 0$, and let us choose $N > \frac{1}{\varepsilon^{1/\alpha}}$.

Then we have for all $n \geq N$:

$\left|\frac{1}{n^\alpha}\right| = \frac{1}{n^\alpha} \leq \frac{1}{N^\alpha} < \varepsilon.$

iii) We show $\lim_{n \to \infty} \sqrt[n]{n} = 1$.

We recall the binomial theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

and define $a_n := \sqrt[n]{n} - 1$, $n \in \mathbb{N}$.

$\Rightarrow \forall n \geq 1: \; a_n \geq 0.$

$\Rightarrow \forall n \geq 2: \quad n = (\sqrt[n]{n})^n = (a_n + 1)^n$

$$= \sum_{k=0}^{n} \binom{n}{k} (a_n)^k \geq 1 + \binom{n}{1} a_n + \binom{n}{2} (a_n)^2$$

$$\geq \binom{n}{2} (a_n)^2 = \frac{n(n-1)}{2} (a_n)^2$$

$\Rightarrow \forall n \geq 2: \quad (a_n)^2 \leq \frac{2}{n-1}$

$\Rightarrow \forall n \geq 2: \quad |a_n| \leq \sqrt{\frac{2}{n-1}} \to 0$ (c.f. example ii)

$\Rightarrow a_n \to 0$ (here we use Thm 4, iii).

iv) For fixed $a > 0$ we have $\lim_{n \to \infty} \sqrt[n]{a} = 1$.

<u>Case 1</u>: $a \geq 1$: $\forall n \geq a: \quad 1 \leq \sqrt[n]{a} \leq \sqrt[n]{n}$

$\Rightarrow$ (Squeeze Thm) $\sqrt[n]{a} \to 1$.

<u>Case 2</u>: $0 < a < 1$: $\frac{1}{a} > 1 \Rightarrow \sqrt[n]{\frac{1}{a}} \to 1$ (case 1)

$\Rightarrow \sqrt[n]{a} = \frac{1}{\sqrt[n]{\frac{1}{a}}} \longrightarrow 1$, by Thm 3, iv).

v) For fixed $x \in \mathbb{R}$ with $|x| < 1$ we have $x^n \to 0$, and more generally, for fixed $k \in \mathbb{N}_0$ we have $n^k x^n \to 0$.

Let $\varepsilon > 0$. We have $|n^k x^n| = n^k |x|^n < \varepsilon$

$\Leftrightarrow |x| < \frac{\varepsilon^{1/n}}{\sqrt[n]{n^k}} = \frac{\varepsilon^{1/n}}{(\sqrt[n]{n})^k}$

We know $\lim_{n \to \infty} \varepsilon^{1/n} = \lim_{n \to \infty} (\sqrt[n]{n})^k = 1$,

$\Rightarrow \exists N \in \mathbb{N}\ \forall n \geq N: \dfrac{\varepsilon^{1/n}}{\sqrt[n]{n^k}} > |x|$.

vi) For fixed $x \in \mathbb{R}$ with $|x| < 1$ we have

$$\lim_{n \to \infty} \sum_{k=0}^{n} x^k = \frac{1}{1-x}:$$

$$\left| \sum_{k=0}^{n} x^k - \frac{1}{1-x} \right| = \left| \frac{1-x^{n+1}}{1-x} - \frac{1}{1-x} \right| = \frac{|x|}{|1-x|} |x|^n \to 0,$$

where we use example iv).

So far, we could only prove convergence of a sequence if we knew the potential limit a priori.
In the following, we will look at principles that allow us to prove convergence without prior knowledge of the limit.

<u>Definition</u>. A sequence (a_n) is called

i) <u>increasing</u> if $a_n \leq a_{n+1}$

ii) <u>strictly increasing</u> if $a_n < a_{n+1}$

iii) <u>decreasing</u> if $a_n \geq a_{n+1}$

iv) <u>strictly decreasing</u> if $a_n > a_{n+1}$

for all its indices n.

A sequence that is increasing or decreasing is called <u>monotonic</u>.

<u>Definition</u>. Let $(a_n)_{n=1}^{\infty}$ be a sequence and let $(n_k)_{k=1}^{\infty}$ be a strictly increasing sequence of natural numbers, i.e., $n_k < n_{k+1}$ for all $k \in \mathbb{N}$ and $n_k \in \mathbb{N}$. Then $(a_{n_k})_{k=1}^{\infty} = (a_{n_1}, a_{n_2}, \dots)$ is called a <u>subsequence</u> of $(a_n)_{n=1}^{\infty}$.

More generally, if (a_n) starts with an index $l \in \mathbb{Z}$, and $(n_k)_{k=1}^{\infty}$ is a strictly increasing sequence with $n_1 \geq l$, then $(a_{n_k})_{k=1}^{\infty}$ is a <u>subsequence</u> of (a_n).

<u>Example</u>. Let $(a_n)_{n=1}^{\infty}$ be defined by $a_n = \frac{1}{n}$, and $(n_k)_{k=1}^{\infty}$ by $n_k = k^2$, then the subsequence $(a_{n_k})_{k=1}^{\infty}$ is given by $a_{n_k} = \frac{1}{k^2}$.

<u>Theorem 5</u>. Every sequence in $\mathbb{R}$ has a monotonic subsequence.

<u>Proof</u>. Let us consider a sequence $(a_n)_{n=1}^{\infty}$.
An index $m \in \mathbb{N}$ of the sequence is called a <u>peak</u> if for all $n > m$ we have $a_n < a_m$.

<u>Case 1</u> There are infinitely many peaks: $m_1 < m_2 < \cdots$

Then $(a_{m_k})_{k=1}^{\infty}$ is a strictly decreasing subsequence of (a_n).

<u>Case 2</u> There are only finitely many peaks.

$\Rightarrow \exists\, n_1 \in \mathbb{N}$ such that n_1 is greater than all peaks

$\Rightarrow \exists\, n_2 > n_1$ such that $a_{n_2} \geq a_{n_1}$.

We can proceed inductively to find a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of natural numbers such that $(a_{n_k})_{k=1}^{\infty}$ is an increasing subsequence. $\square$

<u>Definition.</u> For a sequence of real numbers $(a_n)_{n=k}^{\infty}$ that is bounded above, we define

$$\sup_{n \geq k} a_n := \sup\{a_n : n \geq k\},$$

which is the <u>least upper bound</u> of $(a_n)_{n=k}^{\infty}$.

Likewise, for a sequence of real numbers $(a_n)_{n=k}^{\infty}$ that is bounded below, we define

$$\inf_{n \geq k} a_n := \inf\{a_n : n \geq k\},$$

which is the <u>greatest lower bound</u> of $(a_n)_{n=k}^{\infty}$.

<u>Example</u> For the sequence $a_n := \frac{1}{n}$, $n \geq 1$, we have

$$\inf_{n \geq 1} a_n = 0 \qquad \text{and} \qquad \sup_{n \geq 1} a_n = a_1 = 1.$$

<u>Theorem 6.</u> (Monotone Convergence Theorem)

Every bounded monotonic sequence is convergent. More precisely, if $(a_n)_{n=k}^{\infty}$ is a bounded increasing sequence, then $\lim\limits_{n\to\infty} a_n = \sup\limits_{n\geq k} a_n$.

Likewise, if $(a_n)_{n=k}^{\infty}$ is a bounded decreasing sequence, then $\lim\limits_{n\to\infty} a_n = \inf\limits_{n\geq k} a_n$.

<u>Proof</u>. W.l.o.g. we only prove the statement in the case of a bounded increasing sequence $(a_n)_{n=1}^{\infty}$, and we set $A := \sup\limits_{n\geq 1} a_n$.

We know $\forall n\geq 1 : a_n \leq A$ and

$\forall \varepsilon>0 \; \exists N\in\mathbb{N} : \; A-\varepsilon < a_N$ (A is the smallest upper bound for a_n).

$\Rightarrow$ (a_n is increasing) $\forall \varepsilon>0 \; \exists N\in\mathbb{N} \; \forall n\geq N : \; A-\varepsilon < a_n$

$\Rightarrow \forall \varepsilon>0 \; \exists N\in\mathbb{N} \; \forall n\geq N : \; |A-a_n| = A-a_n < \varepsilon$.

$\Rightarrow \lim\limits_{n\to\infty} a_n = A$ $\square$

<u>Examples</u>. i) The famous "Basel problem" asks for the value of the limit $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{1}{k^2}$ $\left(= \frac{\pi^2}{6}, \text{ Euler } 1734\right)$.

We cannot solve this problem yet, but we can understand that the limit exists: