MTH1032 Analysis I 25/26

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Course structure:

- Year-long module
- Two lectures each week

  One on Tuesdays
  One on Fridays

- One workshop each week

  Group A Mondays
  Group B Wednesdays

Assessment schedule:

100% Continuous Assessment:

- Active participation in workshops: 10%

- Preliminary oral exam: 15%
- Final oral exam: 75%

$$\text{Final grade} = \max\{0.1\,\text{Workshops} + 0.15\,\text{Prelim.} + 0.75\,\text{Final},\ 0.9\,\text{Final}\}.$$

Weekly routine:

- Attend all lectures and all workshops
- Take your own notes
- Once a week, after the lectures, thoroughly revise the material by working through it step by step ("post-paration")
- After the post-paration, do the pre-workshop assignment
- Consistency is key!

Chapter 1. Sequences

We begin by fixing some notation.

$\mathbb{N} = \{1, 2, 3, \dots\}$ "natural numbers"

$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ "integers"

$\mathbb{Q} = \{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}\}$ "rational numbers"

$\mathbb{R}$ = completion of $\mathbb{Q}$ "real numbers"

Definition. A sequence of real numbers $(a_n)_1^\infty$ is a function $f : \mathbb{N} \to \mathbb{R}$, $f(n) = a_n$.

More generally, for any $k \in \mathbb{Z}$, a sequence $(a_n)_{n=k}^\infty$ is a function $f : \{k, k+1, \dots\} \to \mathbb{R}$, $f(n) = a_n$.

Examples. i) The function that lists the natural numbers in natural order is given by

$$f : \mathbb{N} \to \mathbb{R}, \quad f(n) = n.$$

We can express this function as the sequence $(a_n)_1^\infty$ with $a_n = n$ for all $n \geq 1$, i.e.

$$(a_n)_1^\infty = (a_1, a_2, a_3, \dots) = (1, 2, 3, \dots).$$

ii) Let $f: \mathbb{N}_0 \to \mathbb{R}$ be the function that gives the value of an investment after n periods, when the initial investment is P, and the interest rate r is compounded periodically. Then we have

$$f(n) = P(1+r)^n, \quad n \in \mathbb{N}_0.$$

It is more natural to express this in terms of a sequence $(P_n)_{n=0}^{\infty}$ with

$$P_n = P(1+r)^n.$$

We note that we have $P_0 = P(1+r)^0 = P$, where we use the convention $x^0 = 1$ for $x \in \mathbb{R}$.
This example also illustrates the interpretation of the index n of a sequence $(a_n)_{n=1}^{\infty}$ as "time".

<u>Notation</u>: In the following, we use the symbol (a_n) to denote a sequence when we do not want to specify the starting index.

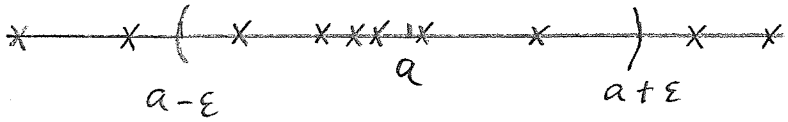
We will now introduce one of the most fundamental concepts in analysis.

<u>Definition</u>. A sequence (a_n) is said to be <u>convergent</u>, if there exists a number $a \in \mathbb{R}$ with the property: for every $\varepsilon > 0$, there exists an integer $N = N(\varepsilon) \in \mathbb{N}$ such that for all $n \geq N$ we have $|a_n - a| < \varepsilon$.

In this case we say: a_n <u>converges to</u> a (as $n \to \infty$), and a is the <u>limit</u> of (a_n).

We write: $\lim\limits_{n \to \infty} a_n = a$ or $a_n \to a$ $(n \to \infty)$.

A sequence that is not convergent is called <u>divergent</u>.



<u>Definition</u>. We define $U_\varepsilon(a) := (a-\varepsilon, a+\varepsilon)$

$= \{x \in \mathbb{R} : a - \varepsilon < x < a + \varepsilon\}$ for $\varepsilon > 0$

and $a \in \mathbb{R}$, and call it <u>ε-neighbourhood</u> of the point a. With this we can write

$$\lim_{n\to\infty} a_n = a \iff \forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq N : a_n \in U_\varepsilon(a).$$

"For every $\varepsilon > 0$, there comes a point in time after which the terms of the sequence stay within the ε-neighbourhood of the point a."

Theorem 1. A sequence (a_n) has at most one limit.

Proof. We prove the theorem by contradiction.

Assume: (a_n) has two limits $a, a' \in \mathbb{R}$ with $a \neq a'$.

We set $\varepsilon := \frac{1}{2}|a-a'|$, then $\varepsilon > 0$ and by the definition of convergence we have

$\exists N_1 \in \mathbb{N} \ \forall n \geq N_1 : |a_n - a| < \varepsilon$,

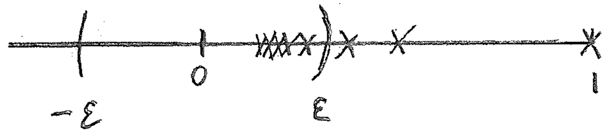
$\exists N_2 \in \mathbb{N} \ \forall n \geq N_2 : |a_n - a'| < \varepsilon$.

Hence, for $n \geq \max\{N_1, N_2\}$, we have

$$2\varepsilon = |a-a'| = |a - a_n + a_n - a'| \leq \underbrace{|a-a_n|}_{<\varepsilon} + \underbrace{|a_n - a'|}_{<\varepsilon} < 2\varepsilon,$$

which is a contradiction.
It follows that our assumption is wrong, and the statement follows. $\square$

Examples. i) We consider the sequence (a_n) defined by $a_n = \frac{1}{n}$, $n \in \mathbb{N}$, so $(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$.

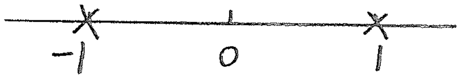


We show $\lim\limits_{n\to\infty} a_n = 0$. Let $\varepsilon > 0$ be a positive number. We want to find $N \in \mathbb{N}$ such that for all $n \geq N$: $|a_n - 0| < \varepsilon$.

We have $|a_n - 0| < \varepsilon \iff \frac{1}{n} < \varepsilon \iff n > \frac{1}{\varepsilon}$.
We choose any $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon}$, then we have for all $n \geq N$:

$$|a_n - 0| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

ii) We consider the sequence (a_n) defined by
$a_n = (-1)^n$, $n \geq 0$.



We show that (a_n) is divergent by contradiction.
__Assume__: (a_n) has a limit $a \in \mathbb{R}$. We set
$\varepsilon := \max\{|1-a|, |-1-a|\} > 0$, then
$|a_n - a| = |(-1)^n - a| = \varepsilon$ for an infinite number of indices n. Hence, for every $N \in \mathbb{N}$ we find an index $n \geq N$ with $|a_n - a| = \varepsilon$, which shows that a cannot be the limit of (a_n) (contradiction).

__Definition.__ A sequence (a_n) is __bounded above__ if there exists a constant $K \in \mathbb{R}$ such that
$a_n \leq K$ for all indices n.
A sequence (a_n) is __bounded below__ if there exists a constant $K \in \mathbb{R}$ such that $a_n \geq K$ for all indices n.

Moreover, a sequence (a_n) is <u>bounded</u> if there exists a constant $K \in \mathbb{R}$ such that $|a_n| \leq K$ for all indices n.

<u>Remark</u>. If (a_n) and (b_n) are bounded sequences, then the sequences (a_n+b_n) and $(a_n b_n)$ are also bounded.

<u>Theorem 2</u>. Every convergent sequence is bounded.

<u>Proof</u>. Let (a_n) be a convergent sequence. W.l.o.g. (without loss of generality), we may assume $(a_n) = (a_n)_{n=1}^{\infty}$, and that $\lim_{n\to\infty} a_n = a$.

$\Rightarrow \exists N \in \mathbb{N}\ \forall n \geq N: \ |a_n - a| < 1.$

$\Rightarrow \forall n \geq N: \ |a_n| = |a_n - a + a| \leq |a_n - a| + |a| < 1 + |a|.$

Hence, we obtain for all $n \geq 1$:

$$|a_n| \leq \max\{|a_1|, |a_2|, \dots, |a_{N-1}|, 1+|a|\}. \qquad \square$$

<u>Remark</u>. The converse of Thm 2 is not true: consider (a_n) given by $a_n = (-1)^n$, $n \geq 0$, then obviously $|a_n| = |(-1)^n| \leq 1$ for all $n \geq 0$, but (a_n) is not convergent.

<u>Theorem 3</u>. Let (a_n) and (b_n) be convergent sequences with limits a and b, respectively.

Then we have

i) $(|a_n|)$ is convergent with limit $|a|$,

ii) (a_n+b_n) is convergent with limit $a+b$,

iii) $(a_n b_n)$ is convergent with limit ab,

iv) if $b \neq 0$, then (c_n) with

$$c_n := \begin{cases} \frac{a_n}{b_n}, & \text{if } b_n \neq 0 \\ 0, & \text{if } b_n = 0 \end{cases}$$

is convergent with limit $\frac{a}{b}$.

Proof. W.l.o.g. we assume $(a_n) = (a_n)_{n=1}^{\infty}$ and $(b_n) = (b_n)_{n=1}^{\infty}$. Let $\varepsilon > 0$.

i) We know $a_n \to a$, so we know

$\exists N_1 \in \mathbb{N} \ \forall n \geq N_1 : |a_n - a| < \varepsilon$.

$\Rightarrow$ (reverse triangle inequality) $\forall n \geq N_1$:

$||a_n| - |a|| \leq |a_n - a| < \varepsilon$.

ii) We know $a_n \to a$, $b_n \to b$, so we know

$\exists N_1 \in \mathbb{N} \ \forall n \geq N_1 : |a_n - a| < \frac{\varepsilon}{2}$

$\exists N_2 \in \mathbb{N} \ \forall n \geq N_2 : |b_n - b| < \frac{\varepsilon}{2}$

$\Rightarrow \forall n \geq \max\{N_1, N_2\} : |a_n + b_n - (a+b)|$

$= |a_n - a + b_n - b| \leq \underbrace{|a_n - a|}_{< \frac{\varepsilon}{2}} + \underbrace{|b_n - b|}_{< \frac{\varepsilon}{2}} < \varepsilon$.

iii) We know $b_n \to b \overset{\text{(Thm 2)}}{\Longrightarrow} \exists K > 0$:

$|b_n| \leq K$ for all $n \geq 1$.

Moreover, we know from $b_n \to b$

$\exists N_1 \in \mathbb{N}\ \forall n \geq N_1: \ |a||b_n - b| < \frac{\varepsilon}{2}$.

Furthermore, $a_n \to a$, so we know

$\exists N_2 \in \mathbb{N}\ \forall n \geq N_2: \ |a_n - a| < \frac{\varepsilon}{2K}$.

$\Longrightarrow \forall n \geq \max\{N_1, N_2\}: \ |a_n b_n - ab|$

$= |(a_n - a)b_n + a(b_n - b)| \leq |a_n - a||b_n| + |a||b_n - b|$

$\leq K|a_n - a| + |a||b_n - b| < \varepsilon$.

iv) We know $b \neq 0$, and by part i) we know $|b_n| \to |b|$.

$\Longrightarrow \exists N_1 \in \mathbb{N}\ \forall n \geq N_1: \ |b_n| \geq \frac{|b|}{2}$.

Moreover, we know $a_n \to a$, so we know

$\exists N_2 \in \mathbb{N}\ \forall n \geq N_2: \ \frac{2}{|b|}|a_n - a| < \frac{\varepsilon}{2}$,

and as $b_n \to b$, we know

$\exists N_3 \in \mathbb{N}\ \forall n \geq N_3: \ \frac{2|a|}{|b|^2}|b_n - b| < \frac{\varepsilon}{2}$.

$\Longrightarrow \forall n \geq \max\{N_1, N_2, N_3\}: \ \left|\frac{a_n}{b_n} - \frac{a}{b}\right| = \frac{|a_n b - a b_n|}{|b_n||b|}$

$\leq \frac{2}{|b|^2}|(a_n - a)b + a(b - b_n)| \leq \frac{2}{|b|}|a_n - a| + \frac{2|a|}{|b|^2}|b - b_n|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. $\square$