Worksheet 9 - Solutions

Q1 1. Let $[a,b]$ be a closed and bounded interval and let $(x_n)_{n=1}^{\infty}$ be a sequence in $[a,b]$.

$\Rightarrow$ (Bolzano-Weierstraß) $\exists$ convergent subsequence $(x_{n_k})_{k=1}^{\infty}$.

We set $x := \lim_{k\to\infty} x_{n_k}$. We still need to show $x \in [a,b]$.

Assume: $x \notin [a,b] \Rightarrow x < a$ or $x > b$

In both cases, we find $\varepsilon > 0$ such that
$U_\varepsilon(x) \cap [a,b] = \emptyset$.

It follows that this ε-neighborhood of x does not contain any terms of (x_{n_k}), which contradicts the fact $x = \lim_{k\to\infty} x_{n_k}$.

$\Rightarrow x \in [a,b]$.

2. Let $(a,b]$ a half-open interval with $a<b$. We choose the sequence

$$x_n := a + \frac{b-a}{n}, \quad n \in \mathbb{N}, \text{ then}$$

$x_n \in (a,b]$ for all $n \in \mathbb{N}$ and converges to a. Hence, every subsequence of (x_n) converges to a, which is not in $(a,b]$. Hence, $(a,b]$ is not compact.

3. Let D be a compact set. If $D = \emptyset$, it is compact. So, let us assume $D \neq \emptyset$. We define $f: D \to \mathbb{R}$, $f(x) = x$, then we know by Thm 34 that f attains its maximum on D, i.e.,

$$M := \sup D = \sup_{x \in D} f(x) = \max_{x \in D} f(x) \in \mathbb{R}.$$

Q2 1. We distinguish several cases.

Case 1: $\xi < -1$ In this case, we find a neighborhood $U_\varepsilon(\xi)$ on which $f(x) = -x$, which is differentiable.

Case 2: $\xi > 1$ In this case, we find a neighborhood $U_\varepsilon(\xi)$ on which $f(x) = x$, which is differentiable.

Case 3: $\xi = 0$ We have for $x \in U_1(0) \setminus \{0\}$

$$\left| \frac{f(x) - f(0)}{x - 0} \right| = \left| \frac{f(x)}{x} \right| \leq \left| \frac{x^2}{x} \right| = |x| \to 0, \quad x \to 0,$$

$$\Rightarrow \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = 0,$$

which shows that f is differentiable at $\xi = 0$.

<u>Case 4:</u> $\xi \in [-1,0) \cup (0,1]$ We already know from Worksheet 8 that f is not continuous at ξ, so it is not differentiable at ξ.

$$\Rightarrow D = (-\infty,-1) \cup \{0\} \cup (1,\infty).$$

2. We have for $x > 0$

$$\frac{f(x)-f(0)}{x-0} = \frac{1}{x}\left(\frac{\sin\sqrt{x}}{\sqrt{x}} - 1\right) = \frac{1}{x}\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(\sqrt{x})^{2n} - 1\right)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{n-1} = -\frac{1}{6} + \underbrace{\sum_{n=2}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{n-1}}_{\to 0,\ x\to 0}$$

$$\Rightarrow f_+'(0) = -\frac{1}{6}.$$

Similarly, we can show that $f_-'(0) = \frac{1}{6}$.

$\Rightarrow$ f is left and right differentiable but not differentiable at $\xi = 0$, as

$$f_+'(0) \neq f_-'(0).$$