Worksheet 7 - Solutions

Q1 We distinguish several cases.

$\alpha < 0$: We observe $\lim_{x \to \infty} x^\alpha = \lim_{x \to \infty} (1+x)^\alpha = 0$

$\Rightarrow \lim_{x \to \infty} f(x) = 0$

$\alpha = 0$: As $f(x) = 0$ we have $\lim_{x \to \infty} f(x) = 0$.

$0 < \alpha < 1$: We have for $x > 0$

$$0 \leq f(x) = x^\alpha \left\{ (1+\tfrac{1}{x})^\alpha - 1 \right\} \leq x^\alpha \frac{\alpha}{x} = \alpha x^{\alpha-1}.$$

As $\alpha - 1 < 0$ we know $\lim_{x \to \infty} x^{\alpha-1} = 0$

$\Rightarrow \lim_{x \to \infty} f(x) = 0$

$\alpha = 1$: As $f(x) = 1$ we have $\lim_{x \to \infty} f(x) = 1$

$\alpha > 1$: We have for $x > 0$

$$f(x) = x^\alpha \left\{ (1+\tfrac{1}{x})^\alpha - 1 \right\} \geq x^\alpha \frac{\alpha}{x} = \alpha x^{\alpha-1}.$$

As $\alpha - 1 > 0$ we know $\lim_{x \to \infty} x^{\alpha-1} = \infty$

$\Rightarrow \lim_{x \to \infty} f(x) = \infty.$

Q2 It is sufficient to find $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ in $\mathbb{R}$ such that $x_n, y_n \to \infty$ but

$$\lim_{n\to\infty} \cos(x_n) \neq \lim_{n\to\infty} \cos(y_n).$$

To this end, we define $x_n = 2n\pi$, $y_n = (2n+1)\pi$, for $n \in \mathbb{N}$, so that $x_n, y_n \to \infty$ and

$$\lim_{n\to\infty} \cos(x_n) = 1 \neq -1 = \lim_{n\to\infty} \cos(y_n).$$

Q3 For $x \neq \pm 1$ we have $|f(x) - f(-2)|$

$$= \left| \frac{3x^2+5x+5}{x^2-1} - \frac{7}{3} \right| = \left| \frac{2x^2+15x+22}{3(x^2-1)} \right|$$

$$= \left| \frac{2x+11}{3(x^2-1)} \right| |x+2|.$$

Moreover, for $x \in U_{\frac{1}{2}}(-2)$ we have

$$\left| \frac{2x+11}{3(x^2-1)} \right| \leq \frac{2|x|+11}{3(|x|^2-1)} \leq \frac{2\cdot\frac{5}{2}+11}{3\left(\left(\frac{3}{2}\right)^2-1\right)} \leq 5.$$

Hence, for any given $\varepsilon > 0$, setting $\delta := \min\left(\frac{1}{2}, \frac{\varepsilon}{5}\right)$, we have for $x \in U_\delta(-2)$

$$|f(x) - f(-2)| < \varepsilon.$$