Worksheet 6 Solutions

Q1: 1) a) We have

$$\Big((-\infty,-1)\cup\{0\}\cup(1/2,1)\cup(1,2]\Big)' = (-\infty,-1]\cup[1/2,2].$$

b) For $D := \{\frac{1}{n} : n\in\mathbb{N}\}$ and $\varepsilon>0$ we have $\quad \mathcal{U}_\varepsilon(0)\cap(D\setminus\{0\}) = \mathcal{U}_\varepsilon(0)\cap D \neq \emptyset$

$\Rightarrow$ 0 is a limit point of D.

However, for any other point $x\neq 0$, we can find an $\varepsilon_0>0$ such that

$$\mathcal{U}_{\varepsilon_0}(x)\cap(D\setminus\{x\}) = \emptyset$$

$\Rightarrow$ $D' = \{0\}$.

2) We want to prove that any real number $x\in\mathbb{R}$ is a limit point of $\mathbb{Q}$.

For $x\in\mathbb{R}$ and $\varepsilon>0$ we can find a $y\in\mathbb{Q}$ such that $\quad x<y<x+\varepsilon$

$\Rightarrow \quad \mathcal{U}_\varepsilon(x)\cap(\mathbb{Q}\setminus\{x\}) \ni y$

$\Rightarrow \quad \mathcal{U}_\varepsilon(x)\cap(\mathbb{Q}\setminus\{x\}) \neq \emptyset.$

Q2 1. We use the geometric sum formula

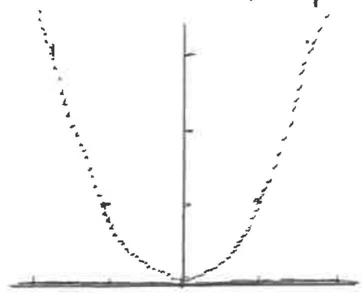
$$t^4 - 1 = (t-1)(1+t+t^2+t^3)$$
$$t^3 - 1 = (t-1)(1+t+t^2)$$

$\Rightarrow$ For $t \neq 1$ we have

$$\frac{t^4-1}{t^3-1} = \frac{1+t+t^2+t^3}{1+t+t^2} \longrightarrow \frac{4}{3} \text{ as } t \to 1,$$

where we use Thm 29 part ii).

2. Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$.



Attempt of a sketch

We show $\lim_{x \to 0} f(x) = 0$. Let $\varepsilon > 0$, then

we have $|f(x) - 0| = f(x) \leq x^2 < \varepsilon$,

for all $x \in U_\delta(0)$ if we set $\delta := \sqrt{\varepsilon}$.

Q3 1. We have for $x \in \mathcal{U}_1(0) \setminus \{0\}$

$$\left| \frac{e^x - 1}{x} - 1 \right| = \left| \frac{1}{x} \left(\sum_{n=0}^{\infty} \frac{1}{n!} x^n - 1 \right) - 1 \right|$$

$$= \left| \sum_{n=1}^{\infty} \frac{1}{n!} x^{n-1} - 1 \right| = \left| \sum_{n=2}^{\infty} \frac{1}{n!} x^{n-1} \right|$$

$$= |x| \left| \sum_{n=2}^{\infty} \frac{1}{n!} x^{n-2} \right| \leq |x| \underbrace{\sum_{n=2}^{\infty} \frac{1}{n!}}_{=:M} = M|x|$$

Hence, for any $\varepsilon > 0$, we have

$$\left| \frac{e^x - 1}{x} - 1 \right| < \varepsilon \quad \text{if} \quad x \in \mathcal{U}_\delta(0) \setminus \{0\}$$

where we define $\delta := \min\left(1, \frac{\varepsilon}{M}\right)$.

2. We have for $x \in \mathcal{U}_1(0) \setminus \{0\}$

$$\left| \frac{1 - \cos x}{x^2} - \frac{1}{2} \right| = \left| \frac{1}{x^2} \left(1 - \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \right) - \frac{1}{2} \right|$$

$$= \left| \frac{1}{x^2} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n}}{(2n)!} - \frac{1}{2} \right| = \left| \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{2n-2}}{(2n)!} - \frac{1}{2} \right|$$

$$= \left| \sum_{n=2}^{\infty} (-1)^{n-1} \frac{x^{2n-2}}{(2n)!} \right| \leq |x| \underbrace{\sum_{n=2}^{\infty} \frac{1}{(2n)!}}_{=:M}$$

Hence, for any $\varepsilon > 0$, we have

$$\left| \frac{1-\cos x}{x^2} - \frac{1}{2} \right| < \varepsilon \quad \text{if} \quad x \in \mathcal{U}_\delta(0) \setminus \{0\}$$

where we define $\delta := \min\left(1, \frac{\varepsilon}{M}\right)$.