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Analysis I (MTH1032)

Worksheet 6

Pre-Workshop Assignment. Understand and memorize

- the concept of limit points of sets.
- the concept of limits of functions (two-sided and one-sided).
- the concept of limits at infinity.

Part 1: Exercises.

Question 1.

1. Find the set of limit points of the following two sets:
   (a) $(-\infty, -1) \cup \{0\} \cup \left(\frac{1}{2}, 1\right) \cup (1, 2]$
   (b) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$
2. Prove $\mathbb{Q}' = \mathbb{R}$.
   Hint: You can use that for every $x \in \mathbb{R}$ and $\epsilon > 0$ we can find a $y \in \mathbb{Q}$ such that $x < y < x + \epsilon$.

Question 2.

1. Find the limit
$$\lim_{t \to 1} \frac{t^4 - 1}{t^3 - 1}.$$
2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by
$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$
   Sketch the graph of f (at least make an attempt) and show
$$\lim_{x \to 0} f(x) = 0.$$

Question 3.

1. Using the series representation of the exponential function, prove that

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1.$$

2. Using the series representation of the cosine, prove that

$$\lim_{x\to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}.$$

Part 2: Exam preparation.

Question 1.

What is a limit point of a given subset of real numbers? How can limit points be characterised in terms of sequences? Can you prove this characterisation?

Question 2.

Give the definition of the limit of a function $f : D \to \mathbb{R}$ at a limit point $\xi \in D'$. What is the definition of the limit at $+\infty$ (if D is not bounded above)? How can we characterise these concepts in terms of sequences?

Question 3.

How are left-hand and right-hand limits of a function defined? What is the relation between one-sided limits and (two-sided) limits? Can you provide an example of a function that has both one-sided limits but no (two-sided) limit at a limit point of its domain?