Worksheet 5 - Solutions

Q1 i) The series is not absolutely convergent by Thm 19 part ii), as we have for $n \geq 1$

$$\frac{n}{3n^2+1} \geq \frac{n}{3n^2+n^2} = \frac{1}{4n},$$

and $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

However, the series is convergent by Thm 16: for $b_n := \frac{n}{3n^2+1}$, $n \geq 1$, we have

$$b_{n+1} \leq b_n \iff \frac{n+1}{3(n+1)^2+1} \leq \frac{n}{3n^2+1}$$

$$\iff (3n^2+1)(n+1) \leq n(3(n+1)^2+1)$$

$$\iff 3n^3+3n^2+n+1 \leq 3n^3+6n^2+4n$$

$$\iff 1 \leq 3n^2+3n,$$

which is true for $n \geq 1$.

ii) The series is absolutely convergent (and hence convergent) by Thm 19 part i), as we have for $n \geq 0$

$$\left| (-1)^n \sqrt{\frac{2^n}{1+4^n}} \right| = \sqrt{\frac{2^n}{1+4^n}} \leq \sqrt{\frac{2^n}{4^n}} = \sqrt{\frac{1}{2^n}}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^n, \text{ and } \sum_{n=0}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^n \text{ is convergent.}$$

Q2 i) We use the root test:

$$\limsup_{k\to\infty} \sqrt[k]{\left(\frac{2k-7}{5k+3}\right)^k} = \limsup_{k\to\infty} \frac{2k-7}{5k+3} = \frac{2}{5} < 1$$

$\Rightarrow$ the series is absolutely convergent.

ii) We use the ratio test: We set $a_\nu := \frac{(-1)^\nu}{\binom{2\nu}{\nu}}$, for $\nu \geq 0$, then we have

$$a_\nu = (-1)^\nu \frac{(\nu!)^2}{(2\nu)!}, \quad \nu \geq 0.$$

$$\Rightarrow \left|\frac{a_{\nu+1}}{a_\nu}\right| = \frac{((\nu+1)!)^2}{(2\nu+2)!} \frac{(2\nu)!}{(\nu!)^2} = \left(\frac{(\nu+1)!}{\nu!}\right)^2 \frac{(2\nu)!}{(2\nu+2)!}$$

$$= (\nu+1)^2 \frac{1}{(2\nu+2)(2\nu+1)} = \frac{\nu^2+2\nu+1}{4\nu^2+6\nu+2}, \quad \nu \geq 0$$

$$\Rightarrow \limsup_{\nu\to\infty} \left|\frac{a_{\nu+1}}{a_\nu}\right| = \lim_{\nu\to\infty} \frac{\nu^2+2\nu+1}{4\nu^2+6\nu+2} = \frac{1}{4} < 1$$

$\Rightarrow$ the series is absolutely convergent.

Q3 We know $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, $|x|<1$, and the Cauchy product of $\sum_{k=0}^{\infty} x^k$ with itself is given by

$$\left(\sum_{k=0}^{\infty} x^k\right)\left(\sum_{k=0}^{\infty} x^k\right) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} x^j x^{k-j}\right)$$

$$= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} x^k\right) = \sum_{k=0}^{\infty}(k+1)x^k$$

$$\Rightarrow \frac{1}{(1-x)^2} = \sum_{k=0}^{\infty}(k+1)x^k, \quad |x|<1.$$

Q4 Let $(a_{\varphi(k)})_{k=0}^{\infty}$ be a rearrangement of the sequence $(a_n)_{n=0}^{\infty}$ with $\lim_{n\to\infty} a_n = a$.

Let $\varepsilon > 0 \Rightarrow \exists N \in \mathbb{N}\ \forall n \geq N: |a_n - a| < \varepsilon$.

Moreover, φ is bijective

$\Rightarrow \exists K \in \mathbb{N}\ \forall k \geq K: \varphi(k) \geq N$

$\Rightarrow \forall k \geq K: |a_{\varphi(k)} - a| < \varepsilon$.