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2025/2026

Analysis I (MTH1032)

Worksheet 5

Pre-Workshop Assignment. Understand and memorize

- the comparison test (all parts)
- the root and the ratio tests and the idea of their proofs
- the definition of the Cauchy product of two series, and under what assumptions it exists
- the concept of rearrangements of sequences and series

Part 1: Exercises.

Question 1.

Classify the series as either absolutely convergent, convergent or divergent.

(i) $$\sum_{n=1}^{\infty}(-1)^n \frac{n}{3n^2+1},$$

(ii) $$\sum_{n=0}^{\infty}(-1)^n \sqrt{\frac{2^n}{1+4^n}}.$$

Question 2.

Test the absolute convergence of the series

(i) $$\sum_{k=1}^{\infty}\left(\frac{2k-7}{5k+3}\right)^k,$$

(ii) $$\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\binom{2\nu}{\nu}}.$$

Question 3.

Consider the Cauchy product of the geometric series with itself to prove the following identity

$$\sum_{n=0}^{\infty}(n+1)x^n = \frac{1}{(1-x)^2}, \qquad |x| < 1.$$

Question 4.

Let $(a_n)_{n=0}^{\infty}$ be a convergent sequence with limit $a \in \mathbb{R}$ and let $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ be a bijection. Show that the rearrangement $(a_{\varphi(k)})_{k=0}^{\infty}$ is convergent and has the same limit a.

Part 2: Exam preparation.

Question 1.

State the comparison test (all parts).

Question 2.

What is the root test, and how does it work? What is the ratio test, and how does it work? Explain the idea shared by both of their proofs.

Question 3.

How is the Cauchy product $\sum_{k=0}^{\infty} c_k$ of two series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ defined? Explain the explicit form of the terms c_k. Under what assumptions does the Cauchy product series converge, and what is its limit?

Question 4.

What is a rearrangement of a sequence? How does it affect convergence of a sequence? And what happens in the case of series?