Worksheet 4 – Solutions

Question 1. We set $b_n := \frac{\sqrt{n}}{n+1}$, $n \geq 1$, then (b_n) is decreasing: $\forall n \geq 1$:

$$b_n \geq b_{n+1} \iff \frac{\sqrt{n}}{n+1} \overset{?}{\geq} \frac{\sqrt{n+1}}{n+2} \iff \sqrt{n}(n+2) \geq (n+1)^{3/2}$$

$$\iff n(n+2)^2 \geq (n+1)^3 \iff n^3 + 4n^2 + 4n \geq n^3 + 3n^2 + 3n + 1$$

$$\iff n^2 + n \geq 1 \quad \checkmark$$

Moreover, (b_n) is convergent with limit 0:

$$0 \leq b_n = \frac{\sqrt{n}}{n+1} \leq \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \to 0, \ n \to \infty.$$

Hence, using Leibniz's test, we can conclude that the series $\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ is convergent.

Question 2. The series is divergent. We show that the sequence $S_n = \sum_{k=1}^{n} (-1)^{k+1} \frac{\sqrt{k} - (-1)^k}{k}$, $n \geq 1$, is divergent. We write

$$S_n = \underbrace{\sum_{k=1}^{n} (-1)^{k+1} \frac{1}{\sqrt{k}}}_{\text{convergent by Leibniz}} + \underbrace{\sum_{k=1}^{n} \frac{1}{k}}_{\text{divergent by condensation}}$$

Assume: (S_n) is convergent.

$$\Rightarrow \quad s_n - \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{\sqrt{k}}, \ n \geq 1, \text{ is convergent}$$

$$\Rightarrow \quad \sum_{n=1}^{n} \frac{1}{k} \text{ is convergent,}$$

which is a contradiction.

<u>Question 3.</u> We have $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \frac{k}{(k+1)!}$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k+1}{(k+1)!} - \frac{1}{(k+1)!} \right) = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{k!} - \frac{1}{(k+1)!} \right)$$

$$= \lim_{n\to\infty} \left(1 - \frac{1}{(n+1)!} \right) = 1$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1.$$

<u>Question 4.</u> The proof runs along the same lines as the proof of Thm 15 (Dirichlet's test).

W.l.o.g. we assume that (b_n) is decreasing.

We know (b_n) is bounded $\Rightarrow \exists M > 0 \ \forall n \geq 0: |b_n| \leq M$.

By assumption $\sum\limits_{k=0}^{\infty} a_k$ is convergent. Let $\varepsilon > 0$.

$$\Rightarrow \exists N \in \mathbb{N} \ \forall m \geq n \geq N: \ \left| \sum_{k=n+1}^{m} a_k \right| < \frac{\varepsilon}{3M}.$$

Using Abel's summation we obtain for $m \geq n \geq N$

$$\left| \sum_{k=n+1}^{m} a_k b_k \right| \leq |b_{m+1}| \left| \sum_{k=n+1}^{m} a_k \right| + \sum_{k=n+1}^{m} \left| \sum_{\nu=n+1}^{k} a_\nu \right| (b_k - b_{k+1})$$

$$< \frac{\varepsilon}{3M} \left(|b_{m+1}| + \sum_{k=n+1}^{m} (b_k - b_{k+1}) \right) = \frac{\varepsilon}{3M} \left(|b_{m+1}| + b_{n+1} - b_{m+1} \right)$$

$$\leq \varepsilon \quad \Rightarrow \text{(Thm 8) convergence.} \quad \square$$