Worksheet 3 - Solutions

Question 1

1) We have for $n \geq m$:

$$\sum_{k=m}^{n} (a_k - a_{k+1}) = \sum_{k=m}^{n} a_k - \sum_{k=m}^{n} a_{k+1}$$

$$= a_m + \sum_{k=m+1}^{n} a_k - a_{n+1} - \sum_{k=m}^{n-1} a_{k+1}$$

$= a_m - a_{n+1}$, where we use that

$$\sum_{k=m+1}^{n} a_k = a_{m+1} + \dots + a_n = \sum_{k=m}^{n-1} a_{k+1} .$$

2) Using part 1 with $a_k = \frac{1}{\sqrt{k}}$, we have for $n \geq 1$:

$$\sum_{k=1}^{n} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right) = 1 - \frac{1}{\sqrt{n+1}} .$$

3) We have for $n \geq 1$

$$\sum_{k=1}^{n} \frac{1}{(k+1)(k+2)} = \sum_{k=1}^{n} \left(\frac{1}{k+1} - \frac{1}{k+2} \right) = \frac{1}{2} - \frac{1}{n+2} .$$

Question 2.

1) To find the cluster points we can choose specific convergent subsequences.

- $n_k = 2k$, $k \geq 1$, gives

$$a_{n_k} = a_{2k} = (-1)^{2k} + \frac{1}{\sqrt{2k}} = 1 + \frac{1}{\sqrt{2k}} \to 1, \ k \to \infty.$$

- $n_k = 2k+1$, $k \geq 1$, gives

$$a_{n_k} = a_{2k+1} = (-1)^{2k+1} + \frac{1}{\sqrt{2k+1}} = -1 + \frac{1}{\sqrt{2k+1}} \to -1, \ k \to \infty.$$

It is not difficult to show that there are no other cluster points, so $C = \{-1, 1\}$.

2) - $n_k = 2k$, $k \geq 1$, gives

$$a_{n_k} = \left(1 + \frac{1}{2k}\right)^{2k} \to e, \ k \to \infty.$$

- $n_k = 2k+1$, $k \geq 1$, gives

$$a_{n_k} = -\left(1 + \frac{1}{2k+1}\right)^{2k+1} \to -e, \ k \to \infty.$$

Again, as it is not difficult to see that there are no other cluster points, we have

$$C = \{e, -e\}.$$