Worksheet 2 – Solutions

Question 1

1) Let (a_n) be a convergent sequence with limit $a \in \mathbb{R}$ and (a_{n_k}) a subsequence.

We want to prove: $\lim_{k\to\infty} a_{n_k} = a$.

Let $\varepsilon > 0$. We know $a_n \to a$, so

$\exists N \in \mathbb{N}\ \forall n \geq N : |a_n - a| < \varepsilon$.

Moreover, as (n_k) is a strictly increasing sequence of natural numbers, we have

$\exists K \in \mathbb{N}\ \forall k \geq K : n_k \geq N$

$\Rightarrow \forall k \geq K : |a_{n_k} - a| < \varepsilon$.

2) We first consider the sequence $a_n := \sqrt[n]{n}$, $n \geq 1$, and the subsequence (a_{n_k}) with $n_k := 2k+1$, $k \geq 1$.

Then $\lim_{k\to\infty} \sqrt[2k+1]{2k+1} = \lim_{k\to\infty} a_{n_k} = \lim_{n\to\infty} a_n = 1$,

where we use example iii following Thm 4.

Next, for $|x| < 1$, we consider $b_n := n x^n$, $n \geq 1$, and the subsequence (b_{n_k}) with $n_k := k!$, $k \geq 1$.

Setting $x = \frac{1}{2}$, we obtain

$$\lim_{k\to\infty} \frac{k!}{2^{k!}} = \lim_{k\to\infty} k! \left(\tfrac{1}{2}\right)^{k!} = \lim_{k\to\infty} k!\, x^{k!}$$

$$= \lim_{k\to\infty} b_{n_k} = \lim_{n\to\infty} b_n = \lim_{n\to\infty} n x^n = 0,$$

where we use example v following Thm 4.

<u>Question 2.</u> 1) Let $(a_n)_{n=1}^{\infty}$ be a convergent sequence with limit $a \in \mathbb{R}$. Let $\varepsilon > 0$, then

$\exists N_1 \in \mathbb{N}\ \forall n \geq N_1 : |a_n - a| < \varepsilon/2.$

Moreover, we have

$\exists N_2 \geq N_1\ \forall n \geq N_2 : \frac{1}{n} \sum_{k=1}^{N_1-1} |a_k - a| < \varepsilon/2.$

$$\Rightarrow \forall n \geq N_2 : \left| \frac{1}{n} \sum_{k=1}^{n} a_k - a \right| = \left| \frac{1}{n} \sum_{k=1}^{n} (a_k - a) \right|$$

$$= \left| \frac{1}{n} \sum_{k=1}^{N_1-1} (a_k - a) + \frac{1}{n} \sum_{k=N_1}^{n} (a_k - a) \right|$$

$$\leq \frac{1}{n} \sum_{k=1}^{N_1-1} |a_k - a| + \frac{1}{n} \sum_{k=N_1}^{n} |a_k - a|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \frac{1}{n} \sum_{k=N_1}^{n} 1 = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \underbrace{\frac{n - N_1 + 1}{n}}_{\leq 1} \leq \varepsilon.$$

2) We consider $b_n := (-1)^{n+1}$, $n \geq 1$, then

$$\frac{1}{n} \sum_{k=1}^{n} b_k = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k+1} = \frac{1}{2n} \{(-1)^{n+1} + 1\} \to 0,\ n \to \infty.$$

$\Rightarrow (b_n)$ is divergent but $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} b_k = 0.$

3) No, for any divergent monotonic sequence (c_n) the limit $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} c_k$ does not exist, because we can show that the sequence $\frac{1}{n}\sum_{k=1}^{n} c_k$ is not bounded. W.l.o.g. let (c_n) be an increasing divergent sequence. By Thm 6, (c_n) cannot be bounded. Let $K > 0$.

$\Rightarrow \exists N_1 \in \mathbb{N}\ \forall n \geq N_1: \ c_n \geq 2K+2.$

Moreover, we know

$\exists N_2 \in \mathbb{N},\ N_2 \geq 2N_1\ \forall n \geq N_2: \frac{1}{n}\sum_{k=1}^{N_1-1} c_k \geq -1.$

$$\Rightarrow \forall n \geq N_2: \ \frac{1}{n}\sum_{k=1}^{n} c_k = \frac{1}{n}\sum_{k=1}^{N_1-1} c_k + \frac{1}{n}\sum_{k=N_1}^{n} c_k$$

$$\geq -1 + (2K+2)\underbrace{\frac{n-N_1+1}{n}}_{\geq \frac{1}{2}} \geq K.$$