Worksheet 1 – Solutions

Question 1. Let $\varepsilon > 0$, then we have

$$|a_n - 1| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n} < \varepsilon$$

$$\Longleftrightarrow \quad n > \frac{1}{\varepsilon}.$$

Hence, we choose any $N > \frac{1}{\varepsilon}$, then we have for all $n \geq N$:

$$|a_n - 1| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

From this, it follows that $a_n \to 1$, $n \to \infty$.

Question 2. We know (a_n) and (b_n) are bounded

$\Rightarrow \exists M_1, M_2 \in \mathbb{R} \;\; \forall n \geq 1: \; |a_n| \leq M_1, \; |b_n| \leq M_2.$

$\Rightarrow \forall n \geq 1: \; |a_n + b_n| \leq |a_n| + |b_n| \leq M_1 + M_2,$

which shows that $(a_n + b_n)$ is bounded, and

$$\forall n \geq 1: \; |a_n b_n| = |a_n||b_n| \leq M_1 M_2,$$

which shows that $(a_n b_n)$ is bounded.

Question 3. i) The sequence (c_k) is bounded below with lower bound -1:

$\forall k \geq 0: \quad c_k = k^2 - k - 1 \geq -1 \iff k(k-1) \geq 0,$

where the last inequality obviously is true.

ii) We show that the sequence (c_k) is not bounded above.

Assume: $\exists M \in \mathbb{R} \ \forall k \geq 0: \ c_k \leq M$.

$\forall k \geq 0: \quad k^2 - 2k + 1 = (k-1)^2 \geq 0 \implies k^2 \geq 2k - 1.$

$\implies \forall k \geq 0: \quad c_k = k^2 - k - 1 \geq 2k - 1 - k - 1 = k - 2$

$\implies \forall k \geq 0: \quad k - 2 \leq c_k \leq M$, which is a contradiction.

iii) The sequence (c_k) cannot be bounded, as this would imply that (c_k) is bounded above.

iv) The sequence (c_k) cannot be convergent, as this would imply boundedness by Thm 2.