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Analysis I (MTH1032)

Worksheet 13 - Solutions

Part 1: Exercises.

Question 1.

1. We use integration by parts (and the Fundamental Theorem of Calculus)

$$\begin{aligned}\int_0^1 x\log(1+x)dx &= \frac{1}{2}x^2\log(1+x)\Big|_0^1 - \int_0^1 \frac{1}{2}x^2\frac{1}{1+x}dx \\ &= \frac{1}{2}\log 2 - \frac{1}{2}\int_0^1 \frac{x^2-1+1}{1+x}dx \\ &= \frac{1}{2}\log 2 - \frac{1}{2}\int_0^1 x-1+\frac{1}{1+x}dx \\ &= \frac{1}{2}\log 2 - \frac{1}{2}\left(\frac{1}{2}-1+\log 2\right) \\ &= \frac{1}{4}.\end{aligned}$$

2. We use the change of variables $x = g(t)$, where $g(t) = \sqrt{t-1}$ is continuously differentiable on $(1,\infty)$ and maps the interval $[2,3]$ bijectively onto $[1,\sqrt{2}]$, so that

$$\begin{aligned}\int_1^{\sqrt{2}} x\sqrt{1+x^2}dx &= \int_2^3 \sqrt{t-1}\sqrt{t}\frac{1}{2\sqrt{t-1}}dt \\ &= \frac{1}{2}\int_2^3 \sqrt{t}dt \\ &= \frac{1}{3}\left(3^{3/2}-2^{3/2}\right).\end{aligned}$$

Question 2.

1. We split the integral into two parts
$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx.$$
In the first integral we use the change of variables $x = -t$ and obtain
$$\int_{-a}^{0} f(x)dx = \int_{a}^{0} f(-t)(-1)dt = \int_{0}^{a} f(-t)dt = -\int_{0}^{a} f(t)dt = -\int_{0}^{a} f(x)dx.$$
Hence, we have
$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = 0.$$

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Hence, we have
$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

Question 3.

We use induction over n to prove $\Gamma(n+1) = \int\limits_0^\infty e^{-t}t^n dt = n!$. We have, using the Fundamental Theorem of Calculus,

$n = 0:$
$$\int\limits_0^\infty e^{-t}dt = \lim_{R\to\infty} \int\limits_0^R e^{-t}dt = \lim_{R\to\infty} \left(1 - e^{-R}\right) = 1$$

$n \to n+1:$ Now we assume that, for some $n \in \mathbb{N}_0$, we have $\int\limits_0^\infty e^{-t}t^n dt = n!$. Using integration by parts we obtain
$$\int\limits_0^\infty e^{-t}t^{n+1}dt = \lim_{R\to\infty} \int\limits_0^R e^{-t}t^{n+1}dt = \lim_{R\to\infty} \left(-e^{-R}R^{n+1} + (n+1)\int_0^R e^{-t}t^n dt\right)$$
$$= (n+1)\int_0^\infty e^{-t}t^n dt = (n+1)n! = (n+1)!,$$
which completes the proof.