Worksheet 12 - Solutions

Q1 From Thm 46, iii) we know $f \in \mathcal{R}([a,b])$,

and from Thm 47, iii) we know

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,dx .$$

Moreover, for every $i \in \{1, \dots, n\}$ we have

$$\int_{x_{i-1}}^{x_i} f(x)\,dx = c_i (x_i - x_{i-1})$$

(see the example after the def. of the Riemann integral).

$$\Rightarrow \int_a^b f(x)\,dx = \sum_{i=1}^{n} c_i (x_i - x_{i-1}) .$$

Q2 We know by Thm 46, i) that $f \in \mathcal{R}([a,b])$, if $f : [a,b] \to \mathbb{R}$,

$f(x) := e^x$.

We define a sequence of partitions P_n of $[a,b]$: For every $n \in \mathbb{N}$, let P_n be given by the points

$$x_i = a + i\,\frac{b-a}{n}, \quad i = 0, \dots, n.$$

As sample points $\xi_1, \dots, \xi_n$ we choose $\xi_i = x_i$, $i = 1, \dots, n$.

Then we have

$$\int_a^b f(x)\,dx = \int_a^b e^x\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} e^{x_i}\,\frac{b-a}{n}$$

$$= \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} e^{a + i\frac{b-a}{n}}$$

$$= \lim_{n\to\infty} e^a\,\frac{b-a}{n} \sum_{i=1}^{n} \left(e^{\frac{b-a}{n}}\right)^i$$

$$= \lim_{n\to\infty} e^a\,\frac{b-a}{n}\, e^{\frac{b-a}{n}}\, \frac{e^{b-a} - 1}{e^{\frac{b-a}{n}} - 1}$$

$$= \lim_{n \to \infty} e^{\frac{b-a}{n}} \frac{\frac{b-a}{n}}{e^{\frac{b-a}{n}} - 1} (e^b - e^a)$$

$$= e^b - e^a.$$

Using the Fundamental Theorem of Calculus, we can compute the integral much easier: $\frac{d}{dx} e^x = e^x$,

so $\int_a^b e^x dx = e^x \Big|_a^b = e^b - e^a.$