Worksheet 11 – Solutions

Q1 1) We have $f(x) = \frac{1}{1-x}$, $x < 1$,

$\Rightarrow f'(x) = \frac{1}{(1-x)^2}$, $f''(x) = \frac{2}{(1-x)^3}$, $f'''(x) = \frac{6}{(1-x)^4}$, ... $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$, $x < 1$, for $n \in \mathbb{N}_0$.

$\Rightarrow f \in C^\infty((-\infty, 1))$ and $f^{(n)}(0) = n!$, $n \in \mathbb{N}_0$.

Hence, the Taylor series of f at $x_0 = 0$ is

$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, x^n = \sum_{n=0}^{\infty} x^n,$$

which is the geometric series. We know already that its radius of convergence is 1, and that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1,$$

so f is real-analytic at $x_0 = 0$.

2) We have $f(x) = \frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)}$

$$= \frac{1}{2}\left(\frac{1}{1-x} + \frac{1}{1+x}\right), \quad |x| < 1.$$

$$\Rightarrow f'(x) = \frac{1}{2}\left(\frac{1}{(1-x)^2} - \frac{1}{(1+x)^2}\right),$$

$$f''(x) = \frac{1}{2}\left(\frac{2}{(1-x)^3} + \frac{2}{(1+x)^3}\right),$$

$$f'''(x) = \frac{1}{2}\left(\frac{6}{(1-x)^4} - \frac{6}{(1+x)^4}\right),$$

$$f^{(n)}(x) = \frac{1}{2}\left(\frac{n!}{(1-x)^{n+1}} + \frac{(-1)^n n!}{(1+x)^{n+1}} \right), \quad |x|<1,$$

for $n \in \mathbb{N}_0$ $\Rightarrow$ $f \in C_\infty((-1,1))$ with

$$f^{(n)}(0) = \frac{1}{2} n! \,(1+(-1)^n) = \begin{cases} n!, & \text{if } n \text{ even} \\ 0, & \text{if } n \text{ odd}. \end{cases}$$

The Taylor series of f at $x_0 = 0$ is

$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, x^n = \sum_{n=0}^{\infty} \frac{1+(-1)^n}{2} x^n = \sum_{n=0}^{\infty} x^{2n}.$$

The radius of convergence is $R =$

$$\frac{1}{\limsup_{n\to\infty} \sqrt[n]{\left|\frac{1+(-1)^n}{2}\right|}} = 1, \quad \text{and for } |x|<1$$

we have

$$\sum_{n=0}^{\infty} \frac{1+(-1)^n}{2} x^n = \frac{1}{2} \sum_{n=0}^{\infty} x^n + \frac{1}{2} \sum_{n=0}^{\infty} (-x)^n$$

$$= \frac{1}{2} \frac{1}{1-x} + \frac{1}{2} \frac{1}{1+x} = f(x),$$

$\Rightarrow$ f is real-analytic at $x_0 = 0$.

3) We have $f(x) = e^x$, so $f^{(n)}(x) = e^x$, $x \in \mathbb{R}$, for all $n \in \mathbb{N}_0$. Hence, we have $f \in C_\infty(\mathbb{R})$, and $f^{(n)}(0) = 1$, $n \in \mathbb{N}_0$.

The Taylor series of e^x at $x_0 = 0$ is

$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n,$$

which shows that the Taylor series of e^x at 0 equals e^x by definition.
It follows that the radius of convergence is ∞, and that e^x is analytic at 0.

4) We have $f(x) = \log\left(\frac{1-x}{1+x}\right) = \log(1-x) - \log(1+x)$, $|x|<1$.

$$\Rightarrow f'(x) = \frac{-1}{1-x} - \frac{1}{1+x} = \frac{-2}{1-x^2}$$

$\Rightarrow$ (comp. 2)) $f \in C_\infty((-1,1))$ with $f(0)=0$, and

$$f^{(n)}(0) = -2\,\frac{(n-1)!}{2}\left(1+(-1)^{n-1}\right) = \begin{cases} 0, & \text{if } n \text{ even} \\ -2\,(n-1)!, & \text{else.} \end{cases}$$

The Taylor series of f at 0 is

$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, x^n = \sum_{n=1}^{\infty} \frac{-1}{n}\left(1+(-1)^{n-1}\right) x^n.$$

The radius of convergence is

$$R = \frac{1}{\limsup\limits_{n\to\infty} \sqrt[n]{\left|\frac{-1}{n}\left(1+(-1)^{n-1}\right)\right|}} = 1,$$ and for $|x|<1$

we have

$$\sum_{n=1}^{\infty} \frac{-1}{n}\left(1+(-1)^{n-1}\right) x^n = \sum_{n=1}^{\infty} \frac{-1}{n} x^n - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(-x)^n - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = \log(1-x) - \log(1+x)$$

$$= \log\left(\frac{1-x}{1+x}\right) = f(x).$$

Hence, f is analytic at 0.

Moreover, as the even coefficients vanish, we have

$$\sum_{n=0}^{\infty} \frac{-1}{n}\left(1+(-1)^{n-1}\right)x^{n} = \sum_{n=0}^{\infty} \frac{-2}{2n+1} x^{2n+1}, \quad |x|<1.$$

Q2 Using Taylor's theorem for $x_0 = 0$, we obtain

$$f(x) = T_n(x) + R_n(x), \text{ where } R_n(x) = 0 \text{ on } \mathbb{R}.$$

$$\Rightarrow f(x) = T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(0)\, x^{k},$$ which is

a polynomial of degree at most n.