Worksheet 10 - Solutions

Q1: We define $h: [a,b] \to \mathbb{R}$, by $h(x) = f(x) - g(x)$. Then h is continuous on $[a,b]$ and differentiable on (a,b) with

$h'(x) = f'(x) - g'(x) = 0$, $x \in (a,b)$.

Hence, for any $x_1 < x_2$ $(x_1, x_2 \in [a,b])$, by the Mean Value Theorem, we obtain

$h(x_2) - h(x_1) = h'(\xi) = 0$. (for some $\xi \in (x_1, x_2)$).

$\Rightarrow$ For all $x_1, x_2 \in [a,b]$: $h(x_1) = h(x_2)$

$\Rightarrow$ h is constant on $[a,b]$

$\Rightarrow$ $f - g$ is constant on $[a,b]$.

Q2 We define $g: (a,b) \to \mathbb{R}$, $g(x) = x - a$, then f and g are differentiable on (a,b) with $g'(x) = 1 \neq 0$ on (a,b), and the limit

$$\lim_{x \to a+} \frac{f'(x)}{g'(x)} = \lim_{x \to a+} f'(x) \text{ exists.}$$

$$\Rightarrow (\text{Thm } 41) \lim_{x \to a+} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a+} \frac{f(x) - f(a)}{g(x)} = \lim_{x \to a+} f'(x).$$

Q3 We prove the statement by induction:

$M=1$: $\quad \frac{d}{dx} f_1(x) = f_1'(x) = \sum_{k=1}^{1} f_k'(x) \underbrace{\prod_{\substack{i=1,\dots,1 \\ i\neq k}} f_i(x)}_{\substack{\text{empty product} \\ =1}}$

$M \to (M+1)$: Suppose we know

$$\frac{d}{dx} \prod_{k=1}^{M} f_k(x) = \sum_{k=1}^{M} f_k'(x) \prod_{\substack{i=1,\dots,M \\ i\neq k}} f_i(x) \qquad (*).$$

Then we have

$$\frac{d}{dx} \prod_{k=1}^{M+1} f_k(x) = \left(\prod_{k=1}^{M} f_k(x)\right)' f_{M+1}(x) + \left(\prod_{k=1}^{M} f_k(x)\right) f_{M+1}'(x)$$

$$\overset{(*)}{=} \sum_{k=1}^{M} f_k'(x) \prod_{\substack{i=1,\dots,M+1 \\ i\neq k}} f_i(x) + f_{M+1}'(x) \prod_{\substack{i=1,\dots,M+1 \\ i\neq M+1}} f_i(x)$$

$$= \sum_{k=1}^{M+1} f_k'(x) \prod_{\substack{i=1,\dots,M+1 \\ i\neq k}} f_i(x).$$

Q4 We use L'Hôpital's Rule:

$$\lim_{x\to 0+} \left(\frac{\cos x}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0+} \frac{x\cos x - \sin x}{x \sin x}$$

$$= \lim_{x\to 0+} \frac{\cos x - x\sin x - \cos x}{\sin x + x\cos x} = \lim_{x\to 0+} \frac{-x\sin x}{\sin x + x\cos x}$$

$$= \lim_{x\to 0+} \frac{-\sin x - x\cos x}{\cos x + \cos x - x\sin x} = 0.$$