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Analysis I (MTH1032)

Worksheet 10

Pre-Workshop Assignment. Understand and memorize

- the theorem on differentiating inverse functions.
- the definition of a local extremum and the necessary condition for the derivative at such a point.
- Rolle's Theorem, and the statement and the proof of the Generalized Mean Value Theorem.
- L'Hôpital's Rule.

Part 1: Exercises.

Question 1.

Let the functions $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $f'(x) = g'(x)$ for all $x \in (a, b)$. Show that f and g differ by a constant.
Hint: Mean Value Theorem.

Question 2.

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $\lim_{x\to a+} f'(x)$ exists. Show that f is right differentiable at $\xi = a$ with derivative

$$f'_+(a) = \lim_{x\to a+} f'(x).$$

Hint: L'Hôpital's rule.

Question 3.

Let $f_k : I \to \mathbb{R}$, for $k = 1, \ldots, n$, be differentiable functions on the interval I. Show the following generalized product rule by mathematical induction:

$$\frac{d}{dx} \prod_{k=1}^{n} f_k(x) = \sum_{k=1}^{n} f_k'(x) \prod_{\substack{i=1,\ldots,n \\ i \neq k}} f_i(x).$$

Question 4.

Find the limit

$$\lim_{x \to 0+} \left(\frac{\cos x}{\sin x} - \frac{1}{x} \right).$$

Hint: L'Hôpital's rule.

Part 2: Exam preparation.

Question 1.

Let $f : I \to \mathbb{R}$ be a bijective function on an interval I. Under which assumptions is the inverse f^{-1} differentiable, and what is its derivative?

Question 2.

Can you state Rolle's Theorem? Explain its geometric meaning using a sketch.

Question 3.

State and prove the Generalized Mean Value Theorem. How does the standard Mean Value Theorem differ? Explain its geometric meaning using a sketch.

Question 4.

State L'Hôpital's Rule and apply it (twice) to find $\lim_{x \to 0+} \frac{1-\cos x}{x^2}$.