be a polynomial of degree $m \in \mathbb{N}$ and $\xi \in \mathbb{R}$.
Then p is continuous at ξ (hence all polynomials are continuous on $\mathbb{R}$): Let $\varepsilon > 0$, then we know from example i) after Thm 29, that for all $x \in U_1(\xi)$: $|p(x) - p(\xi)| \leq M|x - \xi|$, where $M > 0$ is a constant.
Hence, setting $\delta := \min(1, \frac{\varepsilon}{M})$, we have

$$|p(x) - p(\xi)| < \varepsilon \quad \text{for all } x \in U_\delta(\xi).$$

ii) Generally, in order to show that a function $f: D \to \mathbb{R}$ ($D \subset \mathbb{R}$) is not continuous at some given point $\xi \in D$, according to Thm 30, part i), it is sufficient to find a sequence $(x_n)_{n=1}^{\infty}$ in D, converging to ξ, such that

$$\lim_{n \to \infty} f(x_n) \neq f(\xi).$$

Let us apply this to show that the signum function $\operatorname{sign}: \mathbb{R} \to \mathbb{R}$ defined by

$$\operatorname{sign}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

is not continuous at $\xi = 0$.
To this end, we choose $x_n = \frac{1}{n}$, $n \in \mathbb{N}$, then we have

$$\lim_{n \to \infty} \operatorname{sign}(x_n) = 1 \neq 0 = \operatorname{sign}(0).$$

iii) Let $r: D \to \mathbb{R}$ be a rational function, i.e.,
$r(x) = \frac{p(x)}{q(x)}$, where p and q are polynomials and $D = \{x \in \mathbb{R} : q(x) \neq 0\}$ is the "natural" domain of definition.
In light of Thm 30, part ii), and example i) we conclude that r is continuous on D.

iv) The exponential function $\exp: \mathbb{R} \to (0,\infty)$ is continuous on $\mathbb{R}$: By Thm 26, part v), we know that $\lim_{n\to\infty} \exp(x_n) = \exp(\xi)$,
whenever $\xi \in \mathbb{R}$ and $(x_n)_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$ converging to ξ.

v) An important consequence of continuity is the following: Let $f: D \to \mathbb{R}$ $(D \subset \mathbb{R})$ be continuous at the point $\xi \in D$ with $f(\xi) \neq 0$.
Then we can find $\delta > 0$ such that $f(x) \neq 0$ for all $x \in U_\delta(\xi) \cap D$.
To see this, let us choose $\varepsilon := \frac{1}{2}|f(\xi)|$.

$\Rightarrow$ (f cont. at ξ) $\exists\, \delta > 0\ \forall x \in U_\delta(\xi) \cap D$:

$$|f(x) - f(\xi)| < \varepsilon = \tfrac{1}{2}|f(\xi)|.$$

$\Rightarrow$ (reverse triangle) $\frac{1}{2}|f(\xi)| > |f(\xi)| - |f(x)|$, $x \in U_\delta(\xi) \cap D$

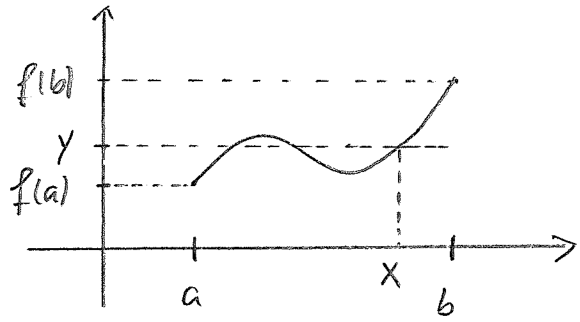
$\Rightarrow$ $|f(x)| > \frac{1}{2}|f(\xi)| > 0$, $x \in U_\delta(\xi) \cap D$.

Theorem 31. (Intermediate Value Theorem)

Let $f: [a,b] \to \mathbb{R}$ be a continuous function on the closed interval $[a,b]$ $(a, b \in \mathbb{R})$. If y is any value between $f(a)$ and $f(b)$, i.e.,

$$y \in [\min(f(a), f(b)), \max(f(a), f(b))],$$ then

there exists a point $x \in [a,b]$ such that $y = f(x)$.



Proof. Case 1: $f(a) = f(b)$ ✓

Case 2: $f(a) \neq f(b)$. W.l.o.g. we may assume that $f(a) < f(b)$. Let y be a value in $(f(a), f(b))$.

We define $M := \{ t \in [a,b] : f(t) \leq y \}$.

$\Rightarrow$ $M \neq \emptyset$ (as $a \in M$), M is bounded below by a and bounded above by b.

$\Rightarrow$ $\sup M \in [a,b]$.

We define $x := \sup M$ and show that $f(x) = y$.

To this end, we choose a sequence $(t_n)_{n=1}^{\infty}$

in M such that $t_n \to x$.

$\Rightarrow$ (Thm 30, part i) $f(x) = \lim_{n\to\infty} \underbrace{f(t_n)}_{\le y} \le y < f(b)$

$\Rightarrow x < b$

Hence, we can choose a sequence $(x_n)_{n=1}^{\infty}$ in $(x, b]$ such that $x_n \to x$.

$\Rightarrow$ (Thm 30, part i) $f(x) = \lim_{n\to\infty} \underbrace{f(x_n)}_{> y} \geq y$

$\Rightarrow y \le f(x) \le y \Rightarrow y = f(x)$. $\square$

<u>Example</u>. Consider the equation $e^x = x + 2$, $x \in \mathbb{R}$. We show that there exists a solution $x_0 \in (1,2)$. To this end, we define $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x - x - 2$.

$\Rightarrow f$ is continuous and $f(1) = e - 3 < 0$,

$f(2) = e^2 - 4 = (e-2)(e+2) > 0$ $(2 < e < 3)$.

$\Rightarrow$ (Thm 31) $\exists\, x_0 \in (1,2)$ such that $f(x_0) = 0$.

For the computation of x_0 we would need numerical methods.

<u>Theorem 32</u>. Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ a continuous function. Then $f(I) = \{f(x) : x \in I\}$ is an interval.

Proof. Let $\alpha, \beta \in f(I)$ such that $\alpha \leq \beta$.

$\Rightarrow \exists\, a, b \in I$ such that $\alpha = f(a)$, $\beta = f(b)$

$\Rightarrow$ (Thm 31) $[\alpha, \beta] \subset f(I) \Rightarrow f(I)$ is an interval. $\square$

Next we deal with inverse functions.

Definition. A function $f: D \to \mathbb{R}$ is called

i) increasing if $f(x_1) \leq f(x_2)$

ii) strictly increasing if $f(x_1) < f(x_2)$

iii) decreasing if $f(x_1) \geq f(x_2)$

iv) strictly decreasing if $f(x_1) > f(x_2)$

for all $x_1, x_2 \in D$ such that $x_1 < x_2$.

In all these cases we call f a (strictly) monotonic function.

Theorem 33. Let $I \subset \mathbb{R}$ be an interval and let $f: I \to \mathbb{R}$ be a continuous and strictly monotonic function. Then $f: I \to J$, where $J := f(I)$, is bijective and its inverse function $f^{-1}: J \to I$ is continuous on J.

Proof. W.l.o.g. we may assume that f is strictly increasing. Then f is injective and, as $J = f(I)$, $f: I \to J$ is surjective.

$\Rightarrow$ the inverse function $f^{-1}: J \to I$ exists.

Now we want to show that f^{-1} is continuous.

Let $y \in J$ and $(y_n)_{n=1}^{\infty}$ a sequence in J with $y_n \to y$.

We set $\xi := f^{-1}(y)$ and $x_n := f^{-1}(y_n)$, $n \in \mathbb{N}$.

Assume: $x_n \not\to \xi \Rightarrow \exists\ \varepsilon > 0$ such that

$|x_n - \xi| \geq \varepsilon$ for infinitely many indices n

$\Rightarrow$ $x_n \geq \xi + \varepsilon$ for infinitely many n

or $x_n \leq \xi - \varepsilon$ for infinitely many n

$\Rightarrow$ (f is strictly increasing)

$y_n = f(x_n) \geq f(\xi + \varepsilon) > f(\xi) = y$ for infinitely many n

or $y_n = f(x_n) \leq f(\xi - \varepsilon) < f(\xi) = y$ for infinitely many n.

$\Rightarrow$ $y_n \not\to y$, which is a contradiction. $\square$

Remark In the situation of Thm 33 it is easy to see that, if f is strictly increasing, then f^{-1} is strictly increasing. Likewise, if f is strictly decreasing, f^{-1} is strictly decreasing as well.

Example. We already know that $\exp: \mathbb{R} \to (0,\infty)$ is continuous and strictly increasing.
Hence, by Thm 33 we know that its inverse function $\log : (0,\infty) \to \mathbb{R}$ exists, and that it is continuous and strictly increasing.

Next we turn to extreme values of continuous functions.

Definition. i) A function $f: D \to \mathbb{R}$ ($D \subset \mathbb{R}$) is bounded above if there exists a $K \in \mathbb{R}$ such that for all $x \in D$: $f(x) \leq K$. Likewise, f is bounded below if there exists a $K \in \mathbb{R}$ such that for all $x \in D$: $f(x) \geq K$. Moreover, f is bounded if there exists a $K \in \mathbb{R}$ such that for all $x \in D$: $|f(x)| \leq K$.

ii) For a function $f: D \to \mathbb{R}$ ($D \subset \mathbb{R}$) we call

$\sup\limits_{x \in D} f(x) := \sup f(D)$ the supremum of f (on D),

$\inf\limits_{x \in D} f(x) := \inf f(D)$ the infimum of f (on D).

Moreover, if they exist, we call

$\max\limits_{x \in D} f(x) := \max f(D)$ the maximum of f (on D),

$\min\limits_{x \in D} f(x) := \min f(D)$ the minimum of f (on D).

Remark. $\max\limits_{x \in D} f(x)$ (or $\min\limits_{x \in D} f(x)$) exists if and

only if there exists $x_0 \in D$ such that for all $x \in D$:
$f(x) \leq f(x_0)$ (or $f(x) \geq f(x_0)$, respectively).

Examples i) The function $f:(0,1) \to \mathbb{R}$, $f(x) = \frac{1}{x}$, is not bounded above as $\lim_{x \to 0+} f(x) = \infty$.

However, for any $\varepsilon > 0$, the function $f:(\varepsilon, 1) \to \mathbb{R}$, $f(x) = \frac{1}{x}$, is bounded above by $f(\varepsilon) = \frac{1}{\varepsilon}$.

ii) Let $D := (0,1]$ and $f: D \to \mathbb{R}$, $f(x) := x^2$.
Then $\max_{x \in D} f(x)$ exists and is given by

$$\max_{x \in D} f(x) = f(1) = 1 = \sup_{x \in D} f(x).$$

Moreover, we have $\inf_{x \in D} f(x) = 0$ but $\min_{x \in D} f(x)$ does not exist.

Definition. A set $D \subset \mathbb{R}$ is called compact if every sequence in D has a convergent subsequence such that its limit is an element of D. In other words:

$\forall$ sequence $(x_n)_{n=1}^{\infty}$ in D $\exists$ subsequence $(x_{n_k})_{k=1}^{\infty}$ such that $\lim_{k \to \infty} x_{n_k} \in D$.

Remark. It is not difficult to show that closed and bounded intervals $[a,b]$, $a,b\in\mathbb{R}$, are compact. However, intervals of the forms (a,b), $[a,b)$, $(a,b]$ or unbounded intervals are not compact.
Moreover, it can be proved that the union of a finite number of compact sets is compact, and that the intersection of an arbitrary number of compact sets is compact.

Theorem 34. Let $D\subset\mathbb{R}$ be a compact set and $f: D\to\mathbb{R}$ be a continuous function. Then $\max_{x\in D} f(x)$ and $\min_{x\in D} f(x)$ exist, i.e., f attains its maximum and its minimum.

Proof. W.l.o.g. we only show the existence of $\max_{x\in D} f(x)$.
We set $s := \sup_{x\in D} f(x)$, which is in $\mathbb{R}$ or ∞.
According to the definition of the supremum, we can find a sequence $(x_n)_{n=1}^{\infty}$ in D such that $f(x_n)\to s$ as $n\to\infty$.
$\Rightarrow$ (D compact) there exists a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$

such that $\lim_{k\to\infty} x_{n_k} = \xi$ with $\xi \in D$.

$\Rightarrow$ (f continuous) $s = \lim_{n\to\infty} f(x_n) = \lim_{k\to\infty} f(x_{n_k})$

$= f(\xi)$

$\Rightarrow$ $s \in \mathbb{R}$ and $f(\xi) = s \geq f(x)$, $x \in D$,

$\Rightarrow$ $\max_{x\in D} f(x)$ exists and is given by $f(\xi)$. $\square$.