Finally we consider the "improper" case.

Definition. Let $f: D \to \mathbb{R}$ $(D \subset \mathbb{R})$ be a function.

i) If $\xi \in D'$, then we say that f tends to infinity as x tends to ξ if

$$\forall K > 0 \; \exists \delta > 0 \; \forall x \in D \setminus \{\xi\} \cap U_\delta(\xi) : f(x) > K$$

We write $\lim_{x \to \xi} f(x) = \infty$ or $f(x) \to \infty$ as $x \to \xi$.

Accordingly, we say that f tends to negative infinity as x tends to ξ if

$$\forall K < 0 \; \exists \delta > 0 \; \forall x \in D \setminus \{\xi\} \cap U_\delta(\xi) : f(x) < K.$$

We write $\lim_{x \to \xi} f(x) = -\infty$ or $f(x) \to -\infty$ as $x \to \xi$.

ii) If $\xi \in (D \cap (-\infty, \xi))'$, then we say that f tends to infinity as x tends to ξ from the left if

$$\forall K > 0 \; \exists \delta > 0 \; \forall x \in D \cap (\xi - \delta, \xi) : f(x) > K$$

We write $\lim_{x \to \xi -} f(x) = \infty$ or $f(x) \to \infty$ as $x \to \xi -$.

The cases $\lim_{x \to \xi -} f(x) = -\infty$, $\lim_{x \to \xi +} f(x) = \infty$

and $\lim_{x \to \xi +} f(x) = -\infty$ are defined analogously.

iii) If D is not bounded above, then we say that <u>f tends to infinity as x tends to infinity</u> if

$$\forall K>0\ \exists L>0\ \forall x\in D\cap(L,\infty):\ f(x)>K.$$

We write $\lim\limits_{x\to\infty} f(x)=\infty$ or $f(x)\to\infty$ as $x\to\infty$.

The cases $\lim\limits_{x\to\infty} f(x)=-\infty$, $\lim\limits_{x\to-\infty} f(x)=\infty$

and $\lim\limits_{x\to-\infty} f(x)=-\infty$ are defined analogously.

<u>Remark.</u> As in Thm 29 part i), for all improper limits in the above definition there exist characterisations in terms of sequences. However, not all statements in Thm 29 part ii) have analogues for improper limits.

<u>Examples.</u> i) We show $\lim\limits_{x\to\infty} \log x=\infty$.

Let $K>0$, then by Thm 27 ii) we know for $x>L:=e^K$: $\log x>\log e^K=K$.

More generally, for any $\alpha>0$, we can show $\lim\limits_{x\to\infty}(\log x)^\alpha=\infty$.

From this, it follows immediately

$$\lim_{x\to\infty} \frac{1}{(\log x)^\alpha} = 0.$$

ii) For any $\alpha, \beta > 0$ we show $\lim\limits_{x\to\infty} \frac{(\log x)^\alpha}{x^\beta} = 0.$

First we observe for any $N \in \mathbb{N}$ with $N > \alpha$ and $x > 1$:

$$x^\beta = e^{\beta \log x} = \sum_{n=0}^{\infty} \frac{1}{n!}(\beta \log x)^n > \frac{1}{N!}\beta^N (\log x)^N.$$

$$\Rightarrow \forall x > 1: \quad 0 \leq \frac{(\log x)^\alpha}{x^\beta} \leq \frac{(\log x)^\alpha}{\frac{1}{N!}\beta^N (\log x)^N}$$

$$= \frac{N!}{\beta^N} \frac{1}{(\log x)^{N-\alpha}} \to 0, \text{ as } x \to \infty,$$

where we used part i) and $N - \alpha > 0$.

From this it follows immediately

$$\lim_{x\to\infty} \frac{(\log x)^\alpha}{x^\beta} = 0.$$

iii) For any $\alpha, \beta > 0$ we show

$$\lim_{x\to\infty} \frac{x^\alpha}{e^{x^\beta}} = 0,$$ where the expression

$$e^{x^\beta} := e^{(x^\beta)}.$$

First we observe for any $N \in \mathbb{N}$ with $N > \frac{\alpha}{\beta}$ and $x > 0$:

$$e^{x^\beta} = \sum_{n=0}^{\infty} \frac{1}{n!} (x^\beta)^n > \frac{1}{N!} (x^\beta)^N = \frac{1}{N!} x^{\beta N}.$$

$\Rightarrow \forall x > 0$:

$$0 \leq \frac{x^\alpha}{e^{x^\beta}} \leq \frac{x^\alpha}{\frac{1}{N!} x^{\beta N}} = \frac{N!}{x^{\beta N - \alpha}} \to 0 \text{ as } x \to \infty,$$

where we used $\beta N - \alpha > 0$ and the easily verifiable fact that $\lim_{x \to \infty} x^\delta = \infty$ for every fixed number $\delta > 0$.

iv) We want to show that $\lim_{x \to 0+} \sin\left(\frac{1}{x}\right)$ does not exist. First we note that in the case of a function $f: (0, \infty) \to \mathbb{R}$, an analogous version of Thm 29 part i) for one-sided limits reads:

$$\lim_{x \to 0+} f(x) = y \Leftrightarrow \forall \text{ sequence } (x_n)_{n=1}^{\infty} \text{ in } (0, \infty) \text{ with } x_n \to 0 \text{ we have } \lim_{n \to \infty} f(x_n) = y.$$

In particular, if $\lim_{x \to 0+} f(x)$ exists, the limit $\lim_{n \to \infty} f(x_n)$ is the same for every possible choice of $(x_n)_{n=1}^{\infty}$ (as long as it is in $(0, \infty)$

and converges to zero).

Hence, if we can find two sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in $(0,\infty)$ with $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = 0$ such that $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$, then we can conclude that $\lim_{x\to 0+} f(x)$ does not exist.

In our concrete case we define

$$x_n := \frac{1}{(2n+\frac{1}{2})\pi}, \quad y_n := \frac{1}{n\pi}, \quad n\in\mathbb{N},$$

then, clearly $x_n \to 0$ and $y_n \to 0$, but

$$\lim_{n\to\infty} \sin\left(\tfrac{1}{x_n}\right) = \lim_{n\to\infty} \sin\left(2\pi n + \tfrac{\pi}{2}\right) = \lim_{n\to\infty} \sin\left(\tfrac{\pi}{2}\right) = 1,$$

$$\lim_{n\to\infty} \sin\left(\tfrac{1}{y_n}\right) = \lim_{n\to\infty} \sin(n\pi) = 0.$$

$\Rightarrow \lim_{x\to 0+} \sin\left(\frac{1}{x}\right)$ does not exist.

Next we introduce the concept of continuity of a function.

<u>Definition</u> Let $f: D \to \mathbb{R}$ be a function and $\xi \in D$. We say that <u>f is continuous at the point ξ</u> if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in U_\delta(\xi) \cap D : |f(x) - f(\xi)| < \varepsilon .$$

We say that <u>f is continuous (on D)</u> if it is continuous at every point of D.

<u>Remark</u> Limits and continuity are closely related: If the point $\xi \in D$ is also a limit point of D, then we can show that

$$f \text{ is continuous at } \xi \iff \lim_{x \to \xi} f(x) = f(\xi) .$$

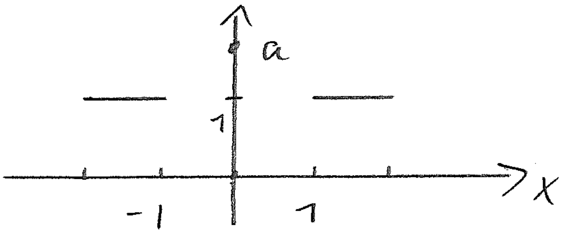
Otherwise, if the point $\xi \in D$ is not a limit point of D, then f is continuous regardless of how the value $f(\xi)$ is defined.

To illustrate this, let us consider the function

$$f : [-2,-1] \cup \{0\} \cup [1,2] \to \mathbb{R}$$

$$f(x) = \begin{cases} 1, & -2 \le x \le -1 \\ a, & x = 0 \\ 1, & 1 \le x \le 2 \end{cases}$$

where $a \in \mathbb{R}$ is some fixed number.



$D := [-2,-1] \cup \{0\} \cup [1,2]$,
$\xi = 0$

For $\delta > 0$ small enough we have $U_\delta(0) \cap D = \{0\}$, so for any given $\varepsilon > 0$, we can choose such a small $\delta > 0$, and then we have

$$\forall x \in \underbrace{U_\delta(0) \cap D}_{= \{0\}} : |f(x) - f(0)| < \varepsilon.$$

$\Rightarrow f$ is continuous at $\xi = 0$.

Moreover, a function can have a limit without being continuous. For example:

$$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

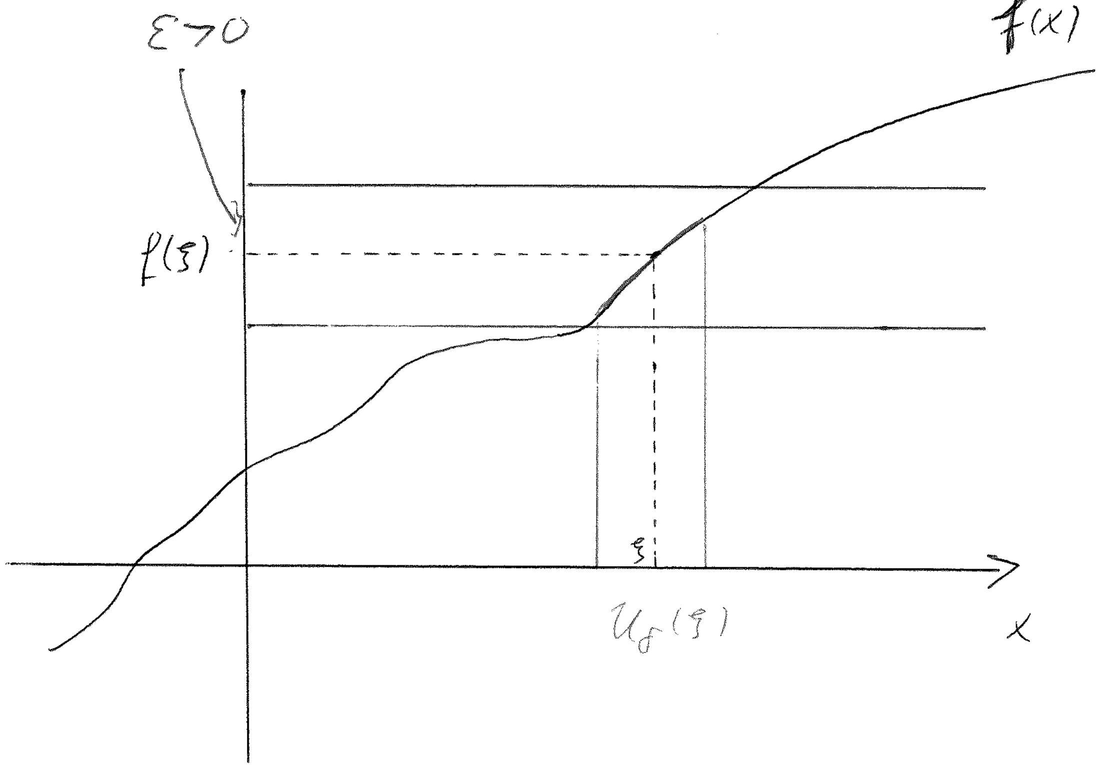
is not continuous at $\xi = 0$ but $\lim\limits_{x \to 0} f(x) = 1$.

Obviously, f is continuous at every point $\xi \neq 0$, and by redefining the value of f at $\xi = 0$ as

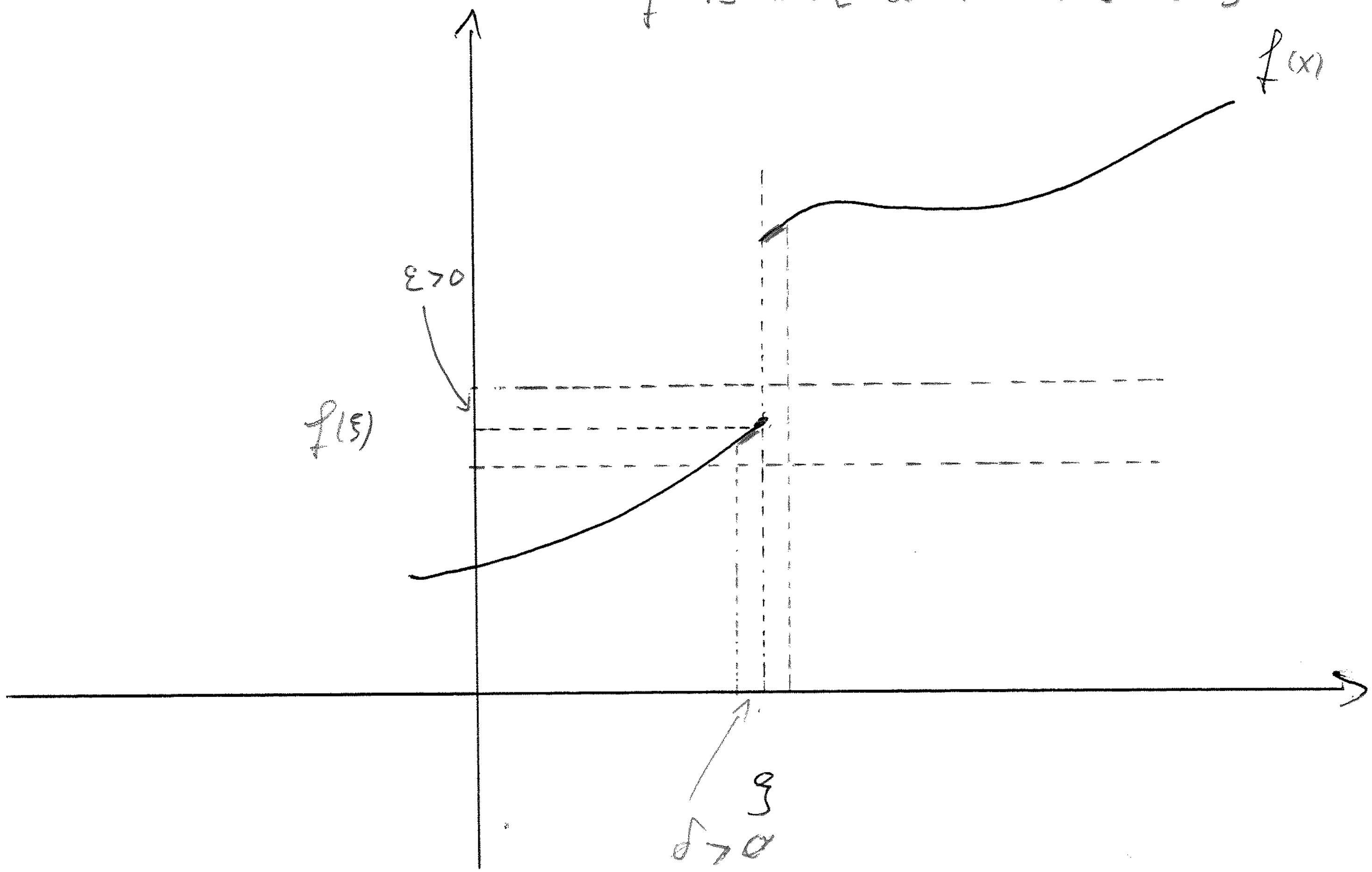
$f(0) := \lim\limits_{x \to 0} f(x) \; (= 1)$, we can make f

continuous on $\mathbb{R}$.

f is continuous at ξ



f is not continuous at ξ



Theorem 30.

i) Let $f: D \to \mathbb{R}$ $(D \subset \mathbb{R})$ be a function and $\xi \in D$.

f is continuous at $\xi \iff \forall$ sequence $(x_n)_{n=1}^{\infty}$ in D with $x_n \to \xi$ we have $\lim_{n\to\infty} f(x_n) = f(\xi)$.

ii) Let $f, g: D \to \mathbb{R}$ $(D \subset \mathbb{R})$ be two functions that are both continuous at the point $\xi \in D$.
Then $f+g$, $f \cdot g$ and, if $g(\xi) \neq 0$, $\frac{f}{g}$

are continuous at ξ.

iii) Let $f: D \to \mathbb{R}$ $(D \subset \mathbb{R})$ be continuous at $\xi \in D$ and let $g: E \to \mathbb{R}$ $(E \subset \mathbb{R})$ be a function such that $f(D) = \{f(x) : x \in D\} \subset E$, and suppose that g is continuous at the point $f(\xi)$.
Then the composition $g \circ f : D \to \mathbb{R}$ defined by $(g \circ f)(x) = g(f(x))$ is continuous at ξ.

__Proof.__ i) "$\Rightarrow$" Let $f: D \to \mathbb{R}$ be continuous at ξ and let $(x_n)_{n=1}^{\infty}$ be a sequence in D with $x_n \to \xi$.
We want to show: $\lim_{n\to\infty} f(x_n) = f(\xi)$.

Let $\varepsilon > 0$, then there exists some $\delta > 0$ such that for all $x \in U_\delta(\xi) \cap D$: $|f(x) - f(\xi)| < \varepsilon$.

As $x_n \to \xi$, we find an $N \in \mathbb{N}$ such that for all $n \geq N$: $x_n \in U_\delta(\xi) \cap D$

$\Rightarrow \forall n \geq N: \; |f(x_n) - f(\xi)| < \varepsilon.$

"$\Leftarrow$" Suppose that for every sequence $(x_n)_{n=1}^\infty$ in D with $x_n \to \xi$ we have $\lim\limits_{n\to\infty} f(x_n) = f(\xi)$.

<u>Assume</u>: f is not continuous at ξ.

$\Rightarrow \exists\, \varepsilon > 0 \; \forall \delta > 0 \; \exists\, x \in U_\delta(\xi) \cap D : |f(x) - f(\xi)| \geq \varepsilon$

$\Rightarrow \forall n = 1, 2, 3, \dots$ we set $\delta = \delta_n = \frac{1}{n}$ and obtain

$x_n \in U_{\frac{1}{n}}(\xi) \cap D$ with $|f(x_n) - f(\xi)| \geq \varepsilon$.

Hence, we have found a sequence $(x_n)_{n=1}^\infty$ in D with $x_n \to \xi$ such that $f(x_n) \not\to f(\xi)$, which is a contradiction.

ii) follows immediately from part i) and Thm 3.

iii) Let $(x_n)_{n=1}^\infty$ be a sequence in D with $x_n \to \xi$.

$\Rightarrow$ (part i, f cont. at ξ) $\lim\limits_{n\to\infty} f(x_n) = f(\xi)$

$\Rightarrow$ (part i, g cont. at $f(\xi)$) $\lim\limits_{n\to\infty} g(f(x_n)) = g(f(\xi))$.

$\Rightarrow$ (part i) $g \circ f$ is continuous at ξ. $\square$

<u>Examples</u>. i) Let $p: \mathbb{R} \to \mathbb{R}$, $p(x) = \sum\limits_{k=0}^{n} a_k x^k$,