Chapter 3. Limits and Continuity

In this chapter we study limits and continuity of functions $f: D \to \mathbb{R}$ with domains $D \subset \mathbb{R}$.

Definition. Let $D \subset \mathbb{R}$ be a subset of the real numbers. A point $x \in \mathbb{R}$ is called a limit point of the set D if for every $\varepsilon > 0$

$$U_\varepsilon(x) \cap (D \setminus \{x\}) \neq \emptyset.$$

The set of limit points of a set D is denoted by D'.

Remark. It is not difficult to show that a point $x \in \mathbb{R}$ is a limit point of the set D if and only if there exists a sequence $(x_n)_{n=1}^{\infty}$ with $x_n \in D \setminus \{x\}$ for all $n \in \mathbb{N}$ such that

$$\lim_{n \to \infty} x_n = x.$$

This explains the name "limit point".

Examples. i) For real numbers $a \leq b$ we have

$$(a,b)' = [a,b], \quad [a,b]' = [a,b],$$

$$[a,b)' = (a,b]' = [a,b].$$

ii) For $D = [a,b] \cup \{c\}$ where $a < b < c$
we have $D' = [a,b]$.

iii) We have $\mathbb{Q}' = \mathbb{R}$.

Definition. Let $f: D \to \mathbb{R}$ $(D \subset \mathbb{R})$ be a function, $\xi \in D'$, and let $y \in \mathbb{R}$.
We say that f tends to the limit y as x tends to ξ if

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in U_\delta(\xi) \cap D \setminus \{\xi\}: \quad |f(x) - y| < \varepsilon.$$

In this case we write

$$\lim_{x \to \xi} f(x) = y \quad \text{or} \quad f(x) \to y \text{ as } x \to \xi.$$

Remark. i) In the definition it is crucial to consider the "deleted" neighbourhood of ξ (i.e. we have to exclude the point ξ). To illustrate this, let us look at the function

$$f: \mathbb{R} \to \mathbb{R}, \quad f(x) := \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Then, according to our definition,

$$\lim_{x \to 0} f(x) = 1, \text{ but } f(0) \neq 1.$$

The limit $\lim_{x\to 0} f(x)$ would not exist if we did not exclude ξ in the above definition.

ii) Similar to the limits of sequences, we can show that if it exists, the limit of a function is unique.

<u>Theorem 29.</u>

i) Let $f: D \to \mathbb{R}$ $(D \subset \mathbb{R})$ be a function, $\xi \in D'$ and $y \in \mathbb{R}$. Then we have

$\lim_{x\to\xi} f(x) = y \iff$ For every sequence $(x_n)_{n=1}^{\infty}$ with $x_n \in D\setminus\{\xi\}$ and $x_n \to \xi$ we have $\lim_{n\to\infty} f(x_n) = y$.

ii) Let $f, g: D \to \mathbb{R}$ $(D \subset \mathbb{R})$ be two functions, $\xi \in D'$, and suppose that $\lim_{x\to\xi} f(x)$ and $\lim_{x\to\xi} g(x)$ exist.

Then, for any $\alpha, \beta \in \mathbb{R}$, we have

$$\lim_{x\to\xi} (\alpha f(x) + \beta g(x)) = \alpha \lim_{x\to\xi} f(x) + \beta \lim_{x\to\xi} g(x),$$

$$\lim_{x\to\xi} (f(x) g(x)) = \lim_{x\to\xi} f(x) \cdot \lim_{x\to\xi} g(x),$$

and if $\lim_{x\to\xi} g(x) \neq 0$ we have

$$\lim_{x\to\xi} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\xi} f(x)}{\lim_{x\to\xi} g(x)} .$$

Proof: i) "$\Rightarrow$" We suppose $\lim_{x\to\xi} f(x) = y$ and choose a sequence $(x_n)_{n=1}^{\infty}$ with $x_n \in D\setminus\{\xi\}$ and $x_n \to \xi$. We show $\lim_{n\to\infty} f(x_n) = y$.

Let $\varepsilon > 0$, then we know:

$\exists \delta > 0 \ \forall x \in D\setminus\{\xi\} \cap U_\delta(\xi): |f(x) - y| < \varepsilon$.

Moreover, we know: $\exists N \in \mathbb{N}: \forall n \geq N: x_n \in U_\delta(\xi)$

$\Rightarrow \forall n \geq N: |f(x_n) - y| < \varepsilon$.

"$\Leftarrow$" We suppose $\lim_{n\to\infty} f(x_n) = y$ for every $(x_n)_{n=1}^{\infty}$ with $x_n \in D\setminus\{\xi\}$ and $x_n \to \xi$.

We show $\lim_{x\to\xi} f(x) = y$, i.e.,

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \in D\setminus\{\xi\} \cap U_\delta(\xi): |f(x) - y| < \varepsilon$.

We show this by contradiction:

Assume: $\exists \varepsilon > 0 \ \forall \delta > 0 \ \exists x \in D\setminus\{\xi\} \cap U_\delta(\xi)$:

$$|f(x) - y| \geq \varepsilon.$$

$\Rightarrow \forall n = 1, 2, 3, \ldots$ we set $\delta = \delta_n = \frac{1}{n}$, so that we obtain $x_n \in D\setminus\{\xi\} \cap U_{1/n}(\xi)$ with

$|f(x_n) - y| \geq \varepsilon$

Hence, we have found a sequence $(x_n)_{n=1}^{\infty}$ with $x_n \in D \setminus \{\xi\}$ and $x_n \to \xi$ such that $f(x_n) \not\to y$, which is a contradiction.

ii) Follows immediately from part i) and Thm 3. $\square$

<u>Examples.</u> i) Let $p: \mathbb{R} \to \mathbb{R}$, $p(x) := \sum_{k=0}^{m} a_k x^k$, be a polynomial of degree $m \in \mathbb{N}$ and $\xi \in \mathbb{R}$.

We show $\lim_{x \to \xi} p(x) = p(\xi)$.

For any $x \in \mathcal{U}_1(\xi)$ we have

$$|p(x) - p(\xi)| = \left| \sum_{k=0}^{m} a_k x^k - \sum_{k=0}^{m} a_k \xi^k \right| \leq \sum_{k=1}^{m} |a_k| |x^k - \xi^k|$$

$$= \sum_{k=1}^{m} |a_k| |x-\xi| \left| \sum_{\nu=0}^{k-1} x^\nu \xi^{k-1-\nu} \right| \leq |x-\xi| \sum_{k=1}^{m} |a_k| \sum_{\nu=0}^{k-1} |x|^\nu |\xi|^{k-1-\nu}$$

$$\underset{|x| \leq 1+|\xi|}{\leq} |x-\xi| \underbrace{\sum_{k=1}^{m} |a_k| \sum_{\nu=0}^{k-1} (1+|\xi|)^\nu |\xi|^{k-1-\nu}}_{=:M} = M|x-\xi|.$$

Hence, for any given $\varepsilon > 0$, setting $\delta := \min\left(1, \frac{\varepsilon}{M}\right)$, we have for $x \in \mathcal{U}_\delta(\xi)$: $|p(x) - p(\xi)| < \varepsilon$.

ii) Let $f: [0,1) \cup (1,\infty) \to \mathbb{R}$, $f(x) := \dfrac{x-1}{\sqrt{x}-1}$, $\xi := 1$.

We show $\lim_{x \to 1} f(x) = 2$.

For any $x \in \mathcal{U}_1(1) \setminus \{1\}$ we have

$$|f(x) - 2| = \left| \frac{x-1}{\sqrt{x}-1} - 2 \right| = \left| \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{\sqrt{x}-1} - 2 \right|$$

$$= |\sqrt{x}+1-2| = |\sqrt{x}-1|$$

$$= \left| \frac{x-1}{\sqrt{x}+1} \right| \leq |x-1| .$$

Hence, for any $\varepsilon > 0$, setting $\delta := \min(\varepsilon, 1)$, we have for $x \in \mathcal{U}_\delta(1) \setminus \{1\}$: $|f(x) - 2| < \varepsilon$.

iii) An analytical definition of sine and cosine is the following:

$$\sin x := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \cos x := \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

for $x \in \mathbb{R}$.

These series representations often are convenient to compute limits, e.g.

$$\lim_{x \to 0} \frac{\sin x}{x}, \quad \lim_{x \to 0} \frac{1 - \cos x}{x^2}, \ldots$$

We show $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

For any $x \in \mathcal{U}_1(0) \setminus \{0\}$ we have

$$\left|\frac{\sin x}{x} - 1\right| = \left|\frac{1}{x}\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} - 1\right|$$

$$= \left|\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n+1)!} - 1\right| = \left|\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n}}{(2n+1)!}\right|$$

$$\leq \sum_{n=1}^{\infty}\frac{|x|^{2n}}{(2n+1)!} = x^2\sum_{n=1}^{\infty}\frac{x^{2n-2}}{(2n+1)!}$$

$$\underset{|x|<1}{\leq} x^2 \underbrace{\sum_{n=1}^{\infty}\frac{1}{(2n+1)!}}_{=:M} = Mx^2 \leq M|x|.$$

Hence, for any given $\varepsilon > 0$, setting $\delta := \min(1, \frac{\varepsilon}{M})$, we have for $x \in U_\delta(0)\setminus\{0\}$

$$\left|\frac{\sin x}{x} - 1\right| \leq M|x| < \varepsilon.$$

Next we consider one-sided limits.

<u>Definition.</u> Let $f: D \to \mathbb{R}$ $(D \subset \mathbb{R})$ be a function and $\xi \in \mathbb{R}$ such that $\xi \in (D \cap (-\infty, \xi))'$
(i.e. $\exists$ sequence $(x_n)_{n=1}^{\infty}$ with $x_n \in D \cap (-\infty, \xi)$ such that $x_n \to \xi$).
We say that <u>f tends to y as x tends to ξ from the left</u> if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in D \cap (\xi - \delta, \xi): |f(x) - y| < \varepsilon.$$

We write $\lim_{x \to \xi -} f(x) = y$ or $f(x) \to y$ as $x \to \xi -$,

and we call this the <u>left-hand limit of f at ξ</u>.

Similarly, if $\xi \in (D \cap (\xi, \infty))'$, then we say that f <u>tends to</u> y <u>as</u> x <u>tends to</u> ξ <u>from the right</u> if

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in D \cap (\xi, \xi + \delta) : |f(x) - y| < \varepsilon .$$

We write $\lim_{x \to \xi +} f(x) = y$ or $f(x) \to y$ as $x \to \xi +$,

and call this the <u>right-hand limit of f at ξ</u>.

<u>Remark</u>. i) For a function $f : D \to \mathbb{R}$ $(D \subset \mathbb{R})$ and $\xi \in \mathbb{R}$ such that $\xi \in (D \cap (-\infty, \xi))' \cap (D \cap (\xi, \infty))'$ it is not difficult to show:

$$\lim_{x \to \xi} f(x) = y \iff \lim_{x \to \xi -} f(x) = \lim_{x \to \xi +} f(x) = y .$$

ii) Using the same ideas, we could prove an analogous version of Thm 29 for one-sided limits.

<u>Examples</u>. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} 0, & x < 0, \\ 1, & 0 \le x < 1, \\ 2, & x = 1, \\ 1, & x > 1 . \end{cases}$$

Then $\lim_{x \to 0-} f(x) = 0$, $\lim_{x \to 0+} f(x) = 1$,

and $\lim_{x \to 0} f(x)$ does not exist.

Moreover, $\lim_{x \to 1-} f(x) = 1$, $\lim_{x \to 1+} f(x) = 1$,

and $\lim_{x \to 1} f(x)$ exists and is equal to 1.

Next we introduce limits at infinity.

<u>Definition.</u> Let $f : D \to \mathbb{R}$ $(D \subset \mathbb{R})$ be a function where D is not bounded above (i.e. $\exists$ sequence $(x_n)_{n=1}^{\infty}$ in D such that $x_n \to \infty$), and let $y \in \mathbb{R}$.
We say that <u>f tends to y as x tends to infinity</u> if $\forall \varepsilon > 0 \; \exists L > 0 \; \forall x \in D \cap (L, \infty) : |f(x) - y| < \varepsilon$.

We write $\lim_{x \to \infty} f(x) = y$ or $f(x) \to y$ as $x \to \infty$.

Accordingly, if $D \subset \mathbb{R}$ is not bounded below, we say that <u>f tends to y as x tends to negative infinity</u> if

$\forall \varepsilon > 0 \; \exists L < 0 \; \forall x \in D \cap (-\infty, L) : |f(x) - y| < \varepsilon$.

We write $\lim_{x \to -\infty} f(x) = y$ or $f(x) \to y$ as $x \to -\infty$.

Remark. i) The convergence of sequences is the case $D = \mathbb{N}$ in the above definition.

ii) Using the same ideas, we could prove an analogous version of Thm 29 for limits at infinity.

Example. Let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(x) := \sqrt{x+1} - \sqrt{x}$. Then we show $\lim_{x \to \infty} f(x) = 0$.

For $x > 0$ we have

$$|f(x) - 0| = |\sqrt{x+1} - \sqrt{x}| = \sqrt{x+1} - \sqrt{x}$$

$$= \frac{(\sqrt{x+1} - \sqrt{x})(\sqrt{x+1} + \sqrt{x})}{\sqrt{x+1} + \sqrt{x}} = \frac{x+1-x}{\sqrt{x+1} + \sqrt{x}} = \frac{1}{\sqrt{x+1} + \sqrt{x}}$$

$$\leq \frac{1}{2\sqrt{x}}.$$

Hence, given any $\varepsilon > 0$, we have

$$\frac{1}{2\sqrt{x}} < \varepsilon \iff x > \frac{1}{4\varepsilon^2}.$$

We set $L := \frac{1}{4\varepsilon^2}$, then we have for $x > L$:

$$|f(x) - 0| < \varepsilon.$$