<u>Theorem 24</u> Let $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ be a power series.

We set $$R := \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}},$$

where we interpret $\frac{1}{\infty}$ as 0, and $\frac{1}{0}$ as ∞ (i.e. $R \in [0,\infty) \cup \{\infty\}$).

If $R = \infty$, then the power series is absolutely convergent for every $x \in \mathbb{R}$.

If $R \in (0,\infty)$, then the power series is

$$\begin{cases} \text{absolutely convergent} & \text{if } |x-x_0| < R \\ \text{divergent} & \text{if } |x-x_0| > R. \end{cases}$$

<u>Proof.</u> We apply the root test. If $R = \infty$, i.e. $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 0$, we have

$$\limsup_{n\to\infty} \sqrt[n]{|a_n||x-x_0|^n} = \limsup_{n\to\infty} \sqrt[n]{|a_n|} \cdot |x-x_0| = 0 < 1,$$

so the power series is abs. convergent for every $x \in \mathbb{R}$.

If $R \in (0,\infty)$, i.e. $\limsup_{n\to\infty} \sqrt[n]{|a_n|} \in (0,\infty)$, then we have

$$\limsup_{n\to\infty} \sqrt[n]{|a_n||x-x_0|^n} = \frac{1}{R}|x-x_0|$$

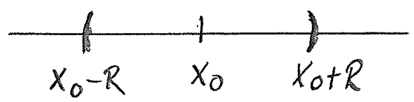
$$\begin{cases} < 1 & \text{if } |x-x_0| < R \Rightarrow \text{abs. convergence} \\ > 1 & \text{if } |x-x_0| > R \Rightarrow \text{divergence} \end{cases}$$

$\square$

Definition. The number R in Thm 24 is called the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$.

Remarks. i) In the case $R=0$ the power series only converges at $x = x_0$. For example, the series $\sum_{n=0}^{\infty} n^n x^n$ has radius of convergence 0.

ii) In the case $R \in (0,\infty)$ the power series is absolutely convergent inside the "disc" $U_R(x_0)$



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and divergent for all $x \in (-\infty, x_0-R) \cup (x_0+R, \infty)$. For the boundary points with $|x-x_0| = R$ there is no general statement on convergence.

iii) An alternative way to compute the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ is using the ratio test, if possible:
If $a_n \neq 0$ for $n \geq N$ ($N \in \mathbb{N}$ fixed) and if $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ exists or is ∞, then for the radius of convergence R we have

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| .$$

It follows, by the way, that for every sequence $(a_n)_{n=0}^{\infty}$ such that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists or is ∞, we have

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| .$$

<u>Examples.</u> i) For $\sum_{n=0}^{\infty} x^n$, i.e. $a_n = 1$, $n \geq 0$, we have $R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} = 1.$

ii) For $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, i.e. $a_n = \frac{1}{n!}$, $n \geq 0$,

we have $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{\frac{1}{n!}}{\frac{1}{(n+1)!}}$

$= \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty,$

so the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ is absolutely convergent for every $x \in \mathbb{R}$.

iii) For $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$, i.e. $a_n = \frac{(-1)^{n+1}}{n}$ for $n \geq 1$ and $a_0 = 0$, we have

$$\limsup_{n\to\infty} \sqrt[n]{\left|\frac{1}{n}\right|} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1, \text{ so}$$

the series converges absolutely for $|x|<1$.

iv) The power series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ and $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ both are absolutely convergent for all $x \in \mathbb{R}$.

Next we study important functions.

<u>Theorem 25</u> For every $x \in \mathbb{R}$ we have

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

<u>Proof</u>. As $\lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, it is sufficient to show $\lim_{n\to\infty} \left\{ \left(1+\frac{x}{n}\right)^n - \sum_{k=0}^{n} \frac{x^k}{k!} \right\} = 0$.

For $n \geq 0$ we have

$$\left| \sum_{k=0}^{n} \frac{x^k}{k!} - \left(1+\frac{x}{n}\right)^n \right| \underset{\text{Binom.}}{=} \left| \sum_{k=0}^{n} \left(\frac{x^k}{k!} - \binom{n}{k} \frac{x^k}{n^k} \right) \right|$$

$$\leq \sum_{k=2}^{n} \frac{|x|^k}{k!} \underbrace{\left(1 - \frac{n(n-1)\cdots(n-k+1)}{n^k}\right)}$$

$$\leq 1 - \frac{(n-k+1)^k}{n^k} = 1 - \left(1 - \frac{k-1}{n}\right)^k$$

$$\underset{\text{Bernoulli}}{\leq} 1 - \left(1 - \frac{k(k-1)}{n}\right) = \frac{k(k-1)}{n}$$

$$\leq \sum_{k=2}^{n} \frac{|x|^k}{k!} \frac{k(k-1)}{n} = \frac{1}{n} \sum_{k=2}^{n} \frac{|x|^k}{(k-2)!}$$

$$\leq \frac{|x|^2}{n} \sum_{k=0}^{\infty} \frac{|x|^k}{k!} \to 0, \quad n \to \infty. \qquad \square$$

<u>Definition</u>. The <u>exponential function</u> $\exp : \mathbb{R} \to \mathbb{R}$ is defined by

$$\exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

The series is called <u>exponential series</u>. Instead of $\exp(x)$ we also write e^x.

<u>Theorem 26.</u> For all $x, y \in \mathbb{R}$ we have

i) $e^{x+y} = e^x e^y$ "functional equation"

ii) $e^x > 0$, $e^{-x} = \frac{1}{e^x}$,

iii) $e^x \geq 1 + x$, $e^x \leq \frac{1}{1-x}$ if $x < 1$,

iv) exp is strictly increasing on $\mathbb{R}$, i.e., $e^x < e^y$ if $x < y$,

v) If $(x_n)_{n=0}^{\infty}$ is a sequence with $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} e^{x_n} = e^x$.

vi) $\exp: \mathbb{R} \to (0,\infty)$ is bijective.

<u>Proof.</u> i) We have for $x, y \in \mathbb{R}$

$$e^x e^y = \left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right)\left(\sum_{k=0}^{\infty} \frac{y^k}{k!}\right) \underset{\text{Thm 22}}{=} \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} \frac{x^j\, y^{k-j}}{j!\,(k-j)!}\right)$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\left(\sum_{j=0}^{k} \binom{k}{j} x^j y^{k-j}\right) = \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!} = e^{x+y}.$$

ii) We have for $x \in \mathbb{R}$: $1 = e^0 = e^{x-x} \underset{i)}{=} e^x e^{-x}$

$$\Rightarrow e^{-x} = \frac{1}{e^x}.$$

Moreover, for $x \geq 0$: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \geq 1 > 0$

and for $x < 0$: $e^x = \frac{1}{e^{-x}} > 0$.

iii) We know for $x \geq -1$ and $n \in \mathbb{N}$

$$\left(1 + \frac{x}{n}\right)^n \underset{\text{Bernoulli}}{\geq} 1 + x \Rightarrow e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \geq 1 + x,$$

and for $x < -1$: $e^x > 0 > 1 + x$.

Moreover, for $x < 1$, we have

$$e^{-x} \geq 1 - x > 0 \Rightarrow e^x \leq \frac{1}{1-x}.$$

iv) For $x < y$ we have

$$e^{y-x} \underset{iii)}{\geq} 1 + y - x > 1 \Rightarrow e^y > e^x.$$

v) If $x_n \to x$, then we know

$\exists N \in \mathbb{N}\ \forall n \geq N: \quad x_n - x < 1.$

$$\Rightarrow \forall n \geq N: \quad e^{x_n} = e^{x_n - x + x} \underset{i)}{=} e^{x_n - x} e^{x}$$

$$\underset{iii)}{\leq} e^{x} \frac{1}{1-(x_n - x)} \to e^{x}, \ n \to \infty,$$

and $e^{x_n} = e^{x} e^{x_n - x} \underset{iii)}{\geq} e^{x}(1 + x_n - x) \to e^{x}, \ n \to \infty.$

$\Rightarrow$ (Squeeze theorem) $e^{x_n} \to e^{x}, \ n \to \infty.$

vi) The injectivity follows from part iv),

so we only need to prove the surjectivity,

i.e. $\forall y \in (0, \infty)\ \exists x \in \mathbb{R}: e^{x} = y.$

Let $y \in (0, \infty)$. We set $S := \{z \in \mathbb{R} : e^{z} \leq y\}$.

It follows from iii) that $\lim_{n \to \infty} e^{-n} = 0$,

so $S \neq \emptyset$. Moreover, for every $z \in S$

$$y \geq e^{z} \geq 1 + z \Rightarrow z \leq y - 1,$$

so S is bounded above.

Hence, $\sup S \in \mathbb{R}$. We define $x := \sup S$,

and we show $e^{x} = y$.

To this end, we observe that

$\exists\, x_n \to x$ with $x_n \in S$ for all $n \geq 1$.

$$\underset{v),\ \text{Thm}\,4}{\Longrightarrow} \quad e^x = \lim_{n\to\infty} e^{x_n} \leq y.$$

<u>Assume</u>. $e^x < y \Rightarrow 1 < e^{-x} y$

We know from iv) and v) that $1 < e^{\frac{1}{n}}$ for all $n \geq 1$ and $e^{\frac{1}{n}} \to e^0 = 1$, $n \to \infty$

$\Rightarrow \exists\, N \in \mathbb{N}\ \forall n \geq N \quad 1 < e^{\frac{1}{n}} < y e^{-x}$

$\Rightarrow \forall n \geq N \quad e^{x + \frac{1}{n}} < y$, which is a contradiction to $x = \sup S$.

Hence, we have $e^x = y$. $\square$

<u>Definition</u>. The inverse function of $\exp : \mathbb{R} \to (0,\infty)$ is called <u>(natural) logarithm</u> and we denote it by $\log : (0,\infty) \to \mathbb{R}$.

<u>Remark</u>. For $x \in \mathbb{R}$ we have $\log(e^x) = x$, and for $x > 0$ we have $e^{\log x} = x$.

<u>Theorem 27</u>. We have for $x, y > 0$

i) $\log(xy) = \log x + \log y$ "functional equation"

ii) log is strictly increasing on $(0,\infty)$, i.e., $\log(x) < \log(y)$ if $x<y$.

<u>Proof</u> i) We show $e^{\log(xy)} = e^{\log x + \log y}$:

$$e^{\log x + \log y} \underset{\text{Thm 26}}{=} e^{\log x} e^{\log y} = xy = e^{\log(xy)}.$$

ii) Let $0<x<y \underset{\text{Thm 26}}{\Rightarrow} \exists\, a,b \in \mathbb{R}$ such that $a<b$ and $x=e^a$, $y=e^b$.

$$\Rightarrow \log x = \log(e^a) = a < b = \log(e^b) = \log y. \quad \square$$

Next we define general powers.

Let $a>0$ and $n\in\mathbb{N}$, then

$$a^n = \underbrace{a\cdot a \cdots a}_{n \text{ times}} = e^{\log a}\cdot e^{\log a} \cdots e^{\log a} \underset{\text{Thm 26}}{=} e^{n\log a}.$$

<u>Definition</u> For $a>0$ and $x\in\mathbb{R}$ we define

$$a^x := e^{x\log a},$$

where a is the <u>base</u> and x is the <u>power</u> or the <u>exponent</u>.

Moreover, the <u>logarithm with base a</u> is defined by

$$\log_a x := \frac{\log x}{\log a}, \quad \text{for } a > 0, a \neq 1 \text{ and } x > 0.$$

<u>Remark</u>. The natural logarithm is the logarithm with base e, and as $\log x$ is the inverse function of e^x, $\log_a x$ is the inverse function of a^x.

<u>Theorem 28.</u> For $a, b > 0$ and $x, y \in \mathbb{R}$ we have

i) $a^x a^y = a^{x+y}$

ii) $(a^x)^y = a^{xy}$

iii) $a^x b^x = (ab)^x$

iv) If $a > 1$, then we have

$$x < y \Rightarrow a^x < a^y,$$

and if $a < 1$, then we have

$$x < y \Rightarrow a^x > a^y.$$

<u>Proof</u> Follows immediately from Thm 26, Thm 27 and the definition of general powers. $\square$