and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is convergent.

ii) The series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ is divergent by part ii) of Thm 19:

$\forall n \geq 1 : \frac{n}{n^2+1} \geq \frac{n}{n^2+n^2} = \frac{1}{2n}$,

and $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

<u>Theorem 20.</u> (Root Test)

Let $(a_n)_{n=0}^{\infty}$ be a sequence.

i) If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then the series $\sum_{n=0}^{\infty} a_n$ is absolutely convergent.

ii) If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1$, then the series $\sum_{n=0}^{\infty} a_n$ is divergent.

<u>Proof.</u> i) We know $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$

$\Rightarrow \exists\, q \in [0,1),\ N \in \mathbb{N}\ \forall n \geq N : \sqrt[n]{|a_n|} \leq q$

$\Rightarrow \forall n \geq N : |a_n| \leq q^n$, and $\sum_{n=0}^{\infty} q^n$ is convergent.

Hence, by Thm 19, $\sum_{n=0}^{\infty} a_n$ is abs. convergent.

ii) We know $\limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1$

$\Rightarrow \sqrt[n]{|a_n|} \geq 1$ for infinitely many n

$\Rightarrow |a_n| \geq 1$ for infinitely many n

$\Rightarrow a_n \not\to 0 \Rightarrow \sum_{n=0}^{\infty} a_n$ is divergent by Thm 12. $\square$

Example. The series $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{-n^2}$ is abs. convergent:

$$\limsup_{n\to\infty} \left(\left(1+\frac{1}{n}\right)^{-n^2}\right)^{\frac{1}{n}} = \limsup_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$$

$$= \limsup_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1.$$

Theorem 21. (Ratio Test)

Let $(a_n)_{n=0}^{\infty}$ be a sequence and $K \in \mathbb{N}$ such that $a_n \neq 0$ for $n \geq K$.

i) If $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then the series $\sum_{n=0}^{\infty} a_n$ is absolutely convergent.

ii) If $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, then the series $\sum_{n=0}^{\infty} a_n$ is divergent.

<u>Proof.</u> i) We know $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

$\Rightarrow \exists q \in [0,1), N \in \mathbb{N} \; \forall n \geq N: \; \left|\frac{a_{n+1}}{a_n}\right| \leq q$

$\Rightarrow \forall n \geq N \quad |a_{n+1}| \leq q|a_n| \leq q^2 |a_{n-1}|$
$\leq \cdots \leq q^{n+1-N} |a_N|$

$\Rightarrow \forall n \geq N+1: \; |a_n| \leq q^n \frac{|a_N|}{q^N}$, and

$\sum_{n=0}^{\infty} q^n$ is convergent

$\Rightarrow$ (Thm 19) $\sum_{n=0}^{\infty} a_n$ is absolutely convergent.

ii) We know $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$

$\Rightarrow \exists N \in \mathbb{N} \; \forall n \geq N: \; \left|\frac{a_{n+1}}{a_n}\right| \geq 1$

$\Rightarrow \forall n \geq N: \; |a_{n+1}| \geq |a_n|$

Together with $a_n \neq 0$ for $n \geq K$, it follows that $a_n \not\to 0$, so $\sum_{n=0}^{\infty} a_n$ is divergent by Thm 12. $\square$

<u>Example.</u> The series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is abs. convergent:

We set $a_n := \frac{n!}{n^n}$, $n \geq 1$, then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!\, n^n}{(n+1)^{n+1}\, n!} = \left(\frac{n}{n+1}\right)^n = \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$\Rightarrow \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} < 1.$$

Remark. As we can see from the series $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$, the root test and the ratio test are inconclusive if $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1$ and $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, respectively.

Next we turn to multiplication of series. Let us first consider finite sums $\sum_{j=0}^{n} a_j$ and $\sum_{k=0}^{n} b_k$. Multiplying them gives

$$S := \left(\sum_{j=0}^{n} a_j\right)\left(\sum_{k=0}^{n} b_k\right) = \sum_{j,k=0}^{n} a_j b_k .$$

We can imagine S as the result of adding all entries of the matrix

$$\begin{pmatrix} a_0b_0 & a_0b_1 & a_0b_2 & \cdots & a_0b_n \\ a_1b_0 & a_1b_1 & a_1b_2 & & \vdots \\ a_2b_0 & a_2b_1 & a_2b_2 & \ddots & \\ \vdots & & & & \\ a_nb_0 & \cdots & & & a_nb_n \end{pmatrix}$$

As the order of summation does not matter we can sum it row-wise: $s = \sum_{j=0}^{m} \sum_{k=0}^{m} a_j b_k$,

column-wise: $s = \sum_{k=0}^{m} \sum_{j=0}^{m} a_j b_k$,

or along diagonals:

$$s = \sum_{k=0}^{2m} \sum_{j=0}^{k} a_j b_{k-j}, \text{ where } a_j := b_k := 0 \text{ if } j,k > m.$$

Definition. For two series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ the series $\sum_{k=0}^{\infty} c_k$, with $c_k = \sum_{j=0}^{k} a_j b_{k-j}$, is called their Cauchy product.

Theorem 22

If $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ are absolutely convergent, then their Cauchy product is absolutely convergent and

$$\sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) = \left(\sum_{j=0}^{\infty} a_j \right) \left(\sum_{k=0}^{\infty} b_k \right).$$

Proof. First we show the absolute convergence of the Cauchy product. To this end it is sufficient to show the boundedness of the partial sums. For $m \geq 0$ we have

$$\sum_{k=0}^{m} |c_k| \leq \sum_{k=0}^{m} \sum_{j=0}^{k} |a_j||b_{k-j}|$$
$$\leq \sum_{j=0}^{m} \sum_{k=0}^{m} |a_j||b_k|$$
$$= \left(\sum_{j=0}^{m} |a_j|\right)\left(\sum_{k=0}^{m} |b_k|\right) \leq \left(\sum_{j=0}^{\infty} |a_j|\right)\left(\sum_{k=0}^{\infty} |b_k|\right).$$

In order to show that $\sum_{k=0}^{\infty} c_k = \left(\sum_{j=0}^{\infty} a_j\right)\left(\sum_{k=0}^{\infty} b_k\right)$,
we show
$$\lim_{m\to\infty} \left(\sum_{k=0}^{2m} c_k - \left(\sum_{j=0}^{m} a_j\right)\left(\sum_{k=0}^{m} b_k\right)\right) = 0.$$

We have for $m \geq 0$

$$\left|\sum_{k=0}^{2m}\left(\sum_{j=0}^{k} a_j b_{k-j}\right) - \sum_{j,k=0}^{m} a_j b_k\right|$$
$$= \left|\sum_{j=0}^{m-1} \sum_{k=m+1}^{2m-j} a_j b_k + \sum_{j=m+1}^{2m} \sum_{k=0}^{2m-j} a_j b_k\right|$$
$$\leq \sum_{j=0}^{m-1} |a_j| \sum_{k=m+1}^{2m-j} |b_k| + \sum_{j=m+1}^{2m} |a_j| \sum_{k=0}^{2m-j} |b_k|$$
$$\leq \sum_{j=0}^{\infty} |a_j| \cdot \underbrace{\sum_{k=m+1}^{\infty} |b_k|}_{\to 0} + \underbrace{\sum_{j=m+1}^{\infty} |a_j|}_{\to 0} \cdot \sum_{k=0}^{\infty} |b_k| \to 0, \; m\to\infty.$$

Here we used $\sum_{k=m+1}^{\infty} |b_k| = \sum_{k=0}^{\infty} |b_k| - \sum_{k=0}^{m} |b_k| \to 0$. □

We now turn to "rearrangements".

<u>Definition.</u> If $(a_n)_{n=0}^{\infty}$ is a sequence and $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ is a bijective mapping, then the sequence $(b_k)_{k=0}^{\infty}$ defined by $b_k = a_{\varphi(k)}$, for $k \geq 0$, is called a <u>rearrangement</u> of (a_n).

It it is not difficult to show that, if $\lim_{n\to\infty} a_n = a$, then we also have $\lim_{k\to\infty} a_{\varphi(k)} = a$.

For series, however, we must be very careful with rearrangements.

The famous <u>Riemann rearrangement theorem</u> tells us that, if $\sum_{n=0}^{\infty} a_n$ is convergent but not absolutely convergent, then for every number $s \in \mathbb{R}$ we can find a rearrangement $(a_{\varphi(k)})_{k=1}^{\infty}$ such that

$$\sum_{k=0}^{\infty} a_{\varphi(k)} = s.$$

This, however, cannot happen with absolutely convergent series.

<u>Theorem 23.</u> Let $\sum_{n=0}^{\infty} a_n$ be absolutely convergent. Then, for every rearrangement $(a_{\varphi(k)})_{k=0}^{\infty}$, the series $\sum_{k=0}^{\infty} a_{\varphi(k)}$ is absolutely convergent and

$$\sum_{k=0}^{\infty} a_{\varphi(k)} = \sum_{n=0}^{\infty} a_n .$$

<u>Proof.</u> Let $\varepsilon > 0 \Rightarrow \exists N \in \mathbb{N} : \sum_{k=N+1}^{\infty} |a_k| < \frac{\varepsilon}{2}$.

Moreover, φ is bijective $\Rightarrow \exists K \in \mathbb{N}$:

$$\{0, 1, \dots, N\} \subset \{\varphi(0), \varphi(1), \dots, \varphi(K)\}$$

$$\Rightarrow \forall k > K : \varphi(k) > N$$

$$\Rightarrow \forall m \geq n > K : \left| \sum_{k=0}^{m} |a_{\varphi(k)}| - \sum_{k=0}^{n} |a_{\varphi(k)}| \right|$$

$$= \sum_{k=n+1}^{m} |a_{\varphi(k)}| \leq \sum_{k=N+1}^{\infty} |a_k| < \frac{\varepsilon}{2} < \varepsilon .$$

$$\Rightarrow \sum_{k=0}^{\infty} a_{\varphi(k)} \text{ is absolutely convergent.}$$

Now let us consider $n \geq K$ $(\geq N)$, then

$$\left| \sum_{k=0}^{n} a_{\varphi(k)} - \sum_{k=0}^{\infty} a_k \right| = \left| \sum_{k=0}^{n} a_{\varphi(k)} - \sum_{k=0}^{N} a_k - \sum_{k=N+1}^{\infty} a_k \right|$$

$$\leq \left| \sum_{\substack{k=0 \\ \varphi(k) > N}}^{n} a_{\varphi(k)} \right| + \sum_{k=N+1}^{\infty} |a_k| \leq 2 \sum_{k=N+1}^{\infty} |a_k| < \varepsilon . \qquad \square .$$

We now turn to the most important class of series.

<u>Definition</u>. Let $(a_n)_{n=0}^{\infty}$ be a sequence and $x_0 \in \mathbb{R}$. A <u>power series</u> is a series of the form

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n , \quad x \in \mathbb{R},$$

where the a_n's are called its <u>coefficients</u> and x_0 its <u>center</u>.

<u>Remarks</u>. i) In many situations we have $x_0 = 0$, and the power series takes the simpler form

$$\sum_{n=0}^{\infty} a_n x^n.$$

ii) The partial sums of a power series

$$S_n(x) = \sum_{k=0}^{n} a_k (x-x_0)^k , \quad n \geq 0,$$

are polynomials in x of degree $\leq n$.

iii) If $a_n = 0$ for all $n > N$ ($N \in \mathbb{N}$ fixed), then the power series is convergent for every $x \in \mathbb{R}$ with

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n = \sum_{n=0}^{N} a_n (x-x_0)^n ,$$

which is a polynomial in x.