<u>Proof.</u> "$\Rightarrow$" Let $f \in \mathcal{R}([a,b])$ and $\varepsilon > 0$ be given. By definition of integrability, we can find partitions P_1 and P_2 of $[a,b]$ such that

$$\int_a^b f(x)\,dx - L(P_1, f) < \frac{\varepsilon}{2}, \quad U(P_2, f) - \int_a^b f(x)\,dx < \frac{\varepsilon}{2}.$$

$\Rightarrow \quad U(P_2, f) - L(P_1, f) < \varepsilon$

$\Rightarrow$ (Thm 44) $U(P^*, f) - L(P^*, f) < \varepsilon$, where P^* is the common refinement of P_1 and P_2.

"$\Leftarrow$" Let $\varepsilon > 0$ be given and let P be a partition of $[a,b]$ such that $U(P,f) - L(P,f) < \varepsilon$.

$$\Rightarrow \quad 0 \leq \int_a^{b*} f(x)\,dx - \int_{a*}^b f(x)\,dx \leq U(P,f) - L(P,f) < \varepsilon$$

As $\varepsilon > 0$ is arbitrary, we have $\int_a^{b*} f(x)\,dx = \int_{a*}^b f(x)\,dx$. □

<u>Theorem 46.</u> i) If $f : [a,b] \to \mathbb{R}$ is continuous, then $f \in \mathcal{R}([a,b])$.

ii) If $f : [a,b] \to \mathbb{R}$ is monotonic, then $f \in \mathcal{R}([a,b])$.

iii) If $f : [a,b] \to \mathbb{R}$ has only finitely many points of discontinuity, then $f \in \mathcal{R}([a,b])$.

iv) Suppose $f \in \mathcal{R}([a,b])$, $m \leq f(x) \leq M$ on $[a,b]$, and $g : [m, M] \to \mathbb{R}$ is continuous. Then for $h : [a,b] \to \mathbb{R}$, $h(x) := g(f(x))$, we have $h \in \mathcal{R}([a,b])$.

<u>Proof.</u> i) First, we observe that a continuous function, defined on a compact set K (here $[a,b]$), is "uniformly" continuous in the following sense:

(*) $\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x, y \in K$ with $|x-y| < \delta$: $|f(x) - f(y)| < \varepsilon$.

To see this, let us assume it is not true; then $\exists \varepsilon > 0 \ \forall n \in \mathbb{N} \ \exists x_n, y_n \in K$ with $|x_n - y_n| \leq \frac{1}{n}$:

$|f(x_n) - f(y_n)| \geq \varepsilon$.

As K is compact we find $(x_{n_k})_{k=1}^{\infty}$ such that

$$p := \lim_{k\to\infty} x_{n_k} \in K \quad \Rightarrow \lim_{k\to\infty} y_{n_k} = p$$

$$\Rightarrow (f \text{ is cont.}) \lim_{k\to\infty} \left(f(x_{n_k}) - f(y_{n_k})\right) = f(p) - f(p) = 0,$$

which is a contradiction.

Now, in order to show that $f \in \mathcal{R}([a,b])$, let $\varepsilon > 0$ be given, and choose $\delta > 0$ according to (*).

We choose a partition P of $[a,b]$ such that

$x_i - x_{i-1} < \delta$, for $i = 1, \dots, n$.

$\Rightarrow$ For $x, y \in [x_{i-1}, x_i]$: $|f(x) - f(y)| < \varepsilon$, $i = 1, \dots, n$,

$$\Rightarrow M_i - m_i = \sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x)$$

$$= \sup_{x, y \in [x_{i-1}, x_i]} \left(f(x) - f(y)\right) \leq \varepsilon, \quad i = 1, \dots, n.$$

$$\Rightarrow U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \leq \varepsilon (b-a).$$

As $\varepsilon > 0$ is arbitrary, $f \in \mathcal{R}([a,b])$ by Thm 45.

ii), iii) and iv) can be proved by similar arguments based on Thm 45. $\square$

Example. Consider $f:[0,1]\to\mathbb{R}$, $f(x)=e^x$, then by Thm 46 i) we know that $f\in\mathcal{R}([0,1])$. We want to compute $\int_0^1 e^x dx$.

We define a sequence of partitions: For $m\in\mathbb{N}$, let P_m be the partition of $[0,1]$ with $x_i=\frac{i}{m}$, $i=0,..,m$.

$$\Rightarrow U(P_m,f)=\sum_{i=1}^m e^{x_i}(x_i-x_{i-1}) = \frac{1}{m}\sum_{i=1}^m (e^{\frac{1}{m}})^i$$

$$=\frac{1}{m}e^{1/m}\frac{e-1}{e^{1/m}-1} = e^{1/m}\frac{1/m}{e^{1/m}-1}(e-1)\to e-1,\ m\to\infty.$$

Similarly, we obtain $L(P_m,f)=\sum_{i=1}^m e^{x_{i-1}}(x_i-x_{i-1})\to e-1$, $m\to\infty$.

$$\Rightarrow \int_0^1 e^x dx = \int_0^{1*} e^x dx \le e-1 \le \int_{0*}^1 e^x dx = \int_0^1 e^x dx$$

$$\Rightarrow \int_0^1 e^x dx = e-1.$$

Theorem 47. i) If $f,g\in\mathcal{R}([a,b])$ and $c\in\mathbb{R}$, then $f+g\in\mathcal{R}([a,b])$, $c\cdot f\in\mathcal{R}([a,b])$, and

$$\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx,$$

$$\int_a^b c\,f(x)\,dx = c\int_a^b f(x)dx.$$

ii) If $f, g \in \mathcal{R}([a,b])$ with $f(x) \le g(x)$ on $[a,b]$, then
$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

iii) If $a<b<c$ and $f \in \mathcal{R}([a,c])$, then $f \in \mathcal{R}([a,b])$ and $f \in \mathcal{R}([b,c])$, and we have
$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)\,dx.$$

iv) If $f \in \mathcal{R}([a,b])$, then $|f| \in \mathcal{R}([a,b])$ and
$$\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx.$$

v) If $f, g \in \mathcal{R}([a,b])$, then $fg \in \mathcal{R}([a,b])$.

<u>Proof.</u> i) We only focus on the sum, so let us assume that $f, g \in \mathcal{R}([a,b])$. By Thm 45, we find partitions P_1 and P_2 of $[a,b]$ such that
$$\mathcal{U}(P_1, f) - L(P_1, f) < \frac{\varepsilon}{2}, \quad \mathcal{U}(P_2, g) - L(P_2, g) < \frac{\varepsilon}{2}$$
$\Rightarrow$ For the refinement $P^* = P_1 \cup P_2$ we have

$$\begin{aligned}
&\mathcal{U}(P^*, f+g) - L(P^*, f+g) \\
&\le \mathcal{U}(P^*, f) + \mathcal{U}(P^*, g) - L(P^*, f) - L(P^*, g) \\
&\le \mathcal{U}(P_1, f) - L(P_1, f) + \mathcal{U}(P_2, g) - L(P_2, g) \\
&< \varepsilon. \quad \Longrightarrow \text{(Thm 45)} \quad f+g \in \mathcal{R}([a,b]).
\end{aligned}$$

Moreover, we have $\int_a^b f(x)+g(x)\,dx \leq U(P^*, f+g)$

$\leq U(P^*, f) + U(P^*, g) \leq U(P_1, f) + U(P_2, g)$

$\leq \int_a^b f(x)dx + \int_a^b g(x)dx + \varepsilon.$

As $\varepsilon > 0$ is arbitrary, this shows that

$$\int_a^b f(x)+g(x)\,dx \leq \int_a^b f(x)dx + \int_a^b g(x)dx.$$

Similarly, we obtain $\int_a^b f(x)+g(x)\,dx \geq \int_a^b f(x)dx + \int_a^b g(x)dx$, hence, we must have equality.

ii), iii) follow using similar arguments.

iv) $|f| \in \mathcal{R}([a,b])$ follows by choosing $g(x) := |x|$ in Thm 46 iii), and the estimate follows from ii).

v) In general, we have

$$fg = \frac{1}{4}\left((f+g)^2 - (f-g)^2\right),$$

so by i) it is sufficient to show $f^2 \in \mathcal{R}([a,b])$ if $f \in \mathcal{R}([a,b])$, which follows by choosing $g(x) := x^2$ in Thm 46 iii). $\square$

The original definition of the Riemann integral was based on so-called Riemann sums.
Let $f \in \mathcal{R}([a,b])$, let P be a partition of $[a,b]$ given by $a = x_0 < x_1 < \dots < x_n = b$, and let

$\xi_1, \xi_2, \dots, \xi_n \in [a,b]$ be given sample points, i.e., $\xi_i \in [x_{i-1}, x_i]$, $i = 1, \dots, n$. Then it is possible to show that for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every partition P on $[a,b]$ with $\max\limits_{i=1,\dots,n} (x_i - x_{i-1}) < \delta$ and for every choice of sample points $\xi_1, \dots, \xi_n$, we have

$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) \right| < \varepsilon.$$

"Riemann sum".

<u>Example</u>. We can use Riemann sums to compute integrals. Let $f: [0,1] \to \mathbb{R}$, $f(x) = x$, then $f \in Q([0,1])$ by Thm 46. For each $n \in \mathbb{N}$, we choose the partition P_n given by $x_i = \frac{i}{n}$, $i = 1, \dots, n$, and we choose the sample points $\xi_i = x_i$, $i = 1, \dots, n$.

Then $\max\limits_{i=1,\dots,n} (x_i - x_{i-1}) = \frac{1}{n} \to 0$, $n \to \infty$, so

$$\lim_{n\to\infty} \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) = \int_0^1 f(x)\,dx.$$

On the other hand, we have

$$\lim_{n\to\infty} \sum_{i=1}^n f(\xi_i)(x_i - x_{i-1}) = \lim_{n\to\infty} \sum_{i=1}^n x_i (x_i - x_{i-1})$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^n i = \lim_{n\to\infty} \frac{n(n+1)}{2n^2} = \frac{1}{2}.$$

Next, we explore the connection between the two fundamental concepts of integration and differentiation.

<u>Definition.</u> Let $I \subset \mathbb{R}$ be an interval. A function $F : I \to \mathbb{R}$ is an <u>antiderivative</u> (or <u>primitive</u>) of $f : I \to \mathbb{R}$ if F is differentiable on I and $F'(x) = f(x)$, $x \in I$.

<u>Remark.</u> Two antiderivatives $F, G : I \to \mathbb{R}$ of the same function f differ only by a constant (Remark iii) after Thm. 39).

<u>Theorem 48.</u> (Fundamental Theorem of Calculus)
Let $f \in \mathcal{R}([a,b])$, $a<b$, and define $F : [a,b] \to \mathbb{R}$ by
$$F(x) = \int_a^x f(t)\,dt.$$

i) If f is continuous at the point $\xi \in [a,b]$, then F is differentiable at ξ with $F'(\xi) = f(\xi)$. In particular, if f is continuous on $[a,b]$, then F is an antiderivative of f.

ii) If $G : [a,b] \to \mathbb{R}$ is an antiderivative of f on $[a,b]$, then $\int_a^b f(x)\,dx = G(b) - G(a)$.

<u>Remarks.</u> i) Thm 48 shows that differentiation and integration are "inverse operations".
Part i) establishes that every continuous function has an antiderivative. This is not true in general for functions $f \in \mathcal{R}([a,b])$. For instance, the function $f: [-1,1] \to \mathbb{R}$, $f(x) = \begin{cases} -1, & -1 \le x \le 0 \\ 1, & \text{else} \end{cases}$,

does not have an antiderivative.
Part ii) shows the importance of antiderivatives for the computation of integrals. This is why we also use the notation $\int f(x)\,dx$ for any antiderivative of f and call it "indefinite integral" of f.
ii) For the difference $G(b)-G(a)$ we use the notation $G(x)\big|_a^b$.

<u>Proof.</u> i) Let f be continuous at $\xi \in [a,b]$, and let $\varepsilon > 0$ be given. Then we choose $\delta > 0$ such that $\forall\, x \in U_\delta(\xi) \cap [a,b]: |f(x) - f(\xi)| < \varepsilon$. Hence, for $x \in U_\delta(\xi) \cap [a,b]$ with $x \neq \xi$, we have

$$\left| \frac{F(x)-F(\xi)}{x-\xi} - f(\xi) \right| = \left| \frac{1}{x-\xi} \int_\xi^x f(t)\,dt - \frac{1}{x-\xi} \int_\xi^x f(\xi)\,dt \right|$$

$$= \left| \frac{1}{x-\xi} \int_\xi^x (f(t)-f(\xi))\,dt \right| \le \frac{1}{x-\xi} \int_\xi^x \underbrace{|f(t)-f(\xi)|}_{<\varepsilon}\,dt$$

$$\le \varepsilon \implies F'(\xi) = f(\xi).$$

ii) Let G be an antiderivative of f on $[a,b]$, and let $\varepsilon > 0$ be given. As $f \in R([a,b])$, we can choose a partition P of $[a,b]$ with $a = x_0 < \dots < x_n = b$ such that

$$\int_a^b f(t)\,dt - \varepsilon < L(P,f), \quad \int_a^b f(t)\,dt + \varepsilon > U(P,f).$$

We apply the mean value theorem to G on $[x_{k-1}, x_k]$ to obtain $G'(x_k) - G'(x_{k-1}) = f(\xi_k)(x_k - x_{k-1})$, for some $\xi_k \in (x_{k-1}, x_k)$, $k = 1, \dots, n$.

$$\Rightarrow G(b) - G(a) = \sum_{k=1}^{n} \left(G'(x_k) - G'(x_{k-1})\right) = \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1})$$

$$\begin{cases} \leq U(P,f) < \int_a^b f(t)\,dt + \varepsilon \\ \geq L(P,f) > \int_a^b f(t)\,dt - \varepsilon \end{cases}$$

$$\Rightarrow \left| G(b) - G'(a) - \int_a^b f(t)\,dt \right| < \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we obtain $\int_a^b f(t)\,dt = G(x)\Big|_a^b$. $\square$

<u>Theorem 49.</u> (Integration by Parts)

Let $f, g \in G([a,b])$, then we have

$$\int_a^b f'(x) g(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f(x) g'(x)\,dx.$$

<u>Proof.</u> We know $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$, $x \in [a,b]$.

$\underset{\text{Thm. 48}}{\Rightarrow}$ $\int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx$

$$= \int_a^b \left(f'(x)g(x) + f(x)g'(x)\right)dx = f(x)g(x)\Big|_a^b . \qquad \square$$

Example. For $x \in \mathbb{R}$ we have

$$\int_0^x t e^t \, dt = t e^t \Big|_0^x - \int_0^x e^t \, dt = x e^x - e^t \Big|_0^x$$
$$= x e^x - e^x + 1.$$

Theorem 50. (Change of Variables)

Let $f : [a,b] \to \mathbb{R}$ be continuous, and let $g : [c,d] \to [a,b]$ be continuously differentiable such that $a = g(c)$, $b = g(d)$. Then

$$\int_a^b f(x)\,dx = \int_c^d f(g(t))\,g'(t)\,dt.$$

Proof. From Thm. 48 we know that f has an antiderivative F on $[a,b]$. Applying the chain rule, we obtain

$$\frac{d}{dt} F(g(t)) = F'(g(t))g'(t) = f(g(t))g'(t), \quad t \in [c,d].$$

$$\underset{\text{Thm 48}}{\Longrightarrow} \int_c^d f(g(t))g'(t)\,dt = F(g(d)) - F(g(c))$$

$$= F(b) - F(a) = \int_a^b f(x)\,dx. \qquad \square$$