f has a derivative of order $n-1$ on I, i.e., $f^{(n-1)}: I \to \mathbb{R}$ exists (we define $f^{(0)} := f$).

If $f^{(n-1)}$ is differentiable at $\xi \in I$, we say that <u>f is n-times differentiable at ξ</u> and call

$f^{(n)}(\xi) := (f^{(n-1)})'(\xi)$ the <u>derivative of order n of f at ξ</u>. If $f^{(n)}(\xi)$ exists at every $\xi \in I$, we say that <u>f is n-times differentiable on I</u> and call $f^{(n)}: I \to \mathbb{R}$ its <u>derivative of order n on I</u>.

Moreover, we define

$$C_n(I) := \{ f : I \to \mathbb{R} : f \text{ is } n\text{-times differentiable on } I \text{ and } f^{(n)} \text{ is continuous on } I \}.$$

"set of n-times continuously differentiable functions on I",

$$C_\infty(I) := \bigcap_{n=0}^{\infty} C_n(I)$$ "set of smooth functions on I".

<u>Remarks</u> i) We have $f^{(0)} = f$, $f^{(1)} = f'$, $f^{(2)} = f''$.

ii) $C_0(I)$ is the set of continuous functions on I.

<u>Definition.</u> Let $f: I \to \mathbb{R}$ n-times differentiable on I, for some $n \in \mathbb{N}_0$, and let $x_0 \in I$. Then

$$T_n(x) := \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k$$

is the <u>n-th-order Taylor polynomial of f centered at x_0</u>.

<u>Theorem 42.</u> (Taylor's Theorem).

Let $f: I \to \mathbb{R}$ be $(n+1)$-times differentiable on I for some $n \in \mathbb{N}_0$, and let $x_0 \in I$ and $p \in \mathbb{N}$. Then for $x \in I$ we have $f(x) = T_n(x) + R_n(x)$, where the remainder is given by

$$R_n(x) = \frac{1}{n!\,p} f^{(n+1)}(x_0 + \vartheta(x-x_0))\,(1-\vartheta)^{n+1-p}\,(x-x_0)^{n+1}$$

for some $\vartheta = \vartheta(x, x_0, n, p) \in (0,1)$.

<u>Proof.</u> Let $x, x_0 \in I$ with $x \neq x_0$, $n \in \mathbb{N}_0$ and $p \in \mathbb{N}$. We define for $t \in I$

$$G(t) := f(x) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(t)(x-t)^k, \quad g(t) := (x-t)^p.$$

$$\Rightarrow G(x) = g(x) = 0, \quad G(x_0) = R_n(x),$$

$$G'(t) = -\frac{1}{n!} f^{(n+1)}(t)(x-t)^n, \quad g'(t) = -p(x-t)^{p-1}.$$

$$\Rightarrow \frac{R_n(x)}{(x-x_0)^p} = \frac{R_n(x)}{g(x_0)} = \frac{-G(x_0)}{-g(x_0)} = \frac{G(x)-G(x_0)}{g(x)-g(x_0)}$$

$$\underset{\text{Thm } 39}{=} \frac{G'(\xi)}{g'(\xi)} \quad \text{for some } \xi \text{ between } x_0 \text{ and } x,$$

which we write as $\xi = x_0 + v(x-x_0)$ for some $v \in (0,1)$.

$$\Rightarrow \frac{R_n(x)}{(x-x_0)^p} = \frac{-f^{(n+1)}(x_0+v(x-x_0))\,(x-x_0-v(x-x_0))^n}{-n!\,p\,(x-x_0-v(x-x_0))^{p-1}}$$

$$= \frac{1}{n!\,p} f^{(n+1)}(x_0+v(x-x_0))\,(1-v)^{n+1-p}(x-x_0)^{n+1-p}. \quad \square$$

As an application of Taylor's Theorem we obtain the following sufficient condition for local extrema.

<u>Theorem 43</u>. Let $f \in C_2((a,b))$ and suppose that $\xi \in (a,b)$ with $f'(\xi) = 0$ and $f''(\xi) \neq 0$.
Then f has a local maximum at ξ if $f''(\xi) < 0$, and f has a local minimum at ξ if $f''(\xi) > 0$.

<u>Proof</u>. Applying Thm 42 with $n=1$, $p=2$, $x_0 = \xi$ to f, gives us

$$f(x) = f(\xi) + f'(\xi)(x-\xi) + \tfrac{1}{2} f''(\xi+v(x-\xi))\,(x-\xi)^2$$
$$= f(\xi) + \tfrac{1}{2} f''(\xi+v(x-\xi))\,(x-\xi)^2 \quad \text{for some } v \in (0,1).$$

As f'' is continuous and $\neq 0$ at ξ, we find a $\delta > 0$ such that $f''(\xi + \vartheta(x-\xi)) < 0$, $x \in U_\delta(\xi)$, if $f''(\xi) < 0$, and $f''(\xi + \vartheta(x-\xi)) > 0$, $x \in U_\delta(\xi)$, if $f''(\xi) > 0$. $\square$

Another application of Taylor's Theorem leads to power series representations of C_∞-functions.

<u>Definition</u>. Let $f \in C_\infty((a,b))$, where $-\infty \le a < b \le \infty$, and let $x_0 \in (a,b)$. The power series

$$\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k$$

is called the <u>Taylor series of f centered at x_0</u>. Moreover, the function f is <u>real-analytic</u> at x_0 if there exists a $\delta > 0$ such that

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(x_0)(x-x_0)^k, \quad x \in U_\delta(x_0).$$

<u>Remark</u>. A function $f \in C_\infty((a,b))$ is real-analytic at $x_0 \in (a,b) \iff f(x) = \lim_{n\to\infty} T_n(x)$ on $U_\delta(x_0)$, where $T_n(x)$ is the Taylor polynomial of n-th order of f centered at x_0.

<u>Examples</u>. i) Not every smooth function is real-analytic: Consider $f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0, \end{cases} \quad \text{and } x_0 = 0.$$

It is not difficult to show that $f \in C_\infty(\mathbb{R})$ with $f^{(k)}(0) = 0$ for all $k \in \mathbb{N}_0$, so that

$$\sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(0)\, x^k = 0 \quad \text{for all } x \in \mathbb{R}, \text{ while}$$

$f(x) \neq 0$ for $x \neq 0$.

ii) Let $f: (-1, \infty) \to \mathbb{R}$, $f(x) = \log(1+x)$, and $x_0 = 0$.
Then $f'(x) = \frac{1}{1+x}$, $f''(x) = \frac{-1}{(1+x)^2}$, $f^{(3)}(x) = \frac{(-1)^2\, 2}{(1+x)^3}$.

$$\Rightarrow f^{(k)}(x) = \frac{(-1)^{k-1}(k-1)!}{(1+x)^k}, \quad k \geqslant 0, \quad \Rightarrow f \in C_\infty((-1,\infty))$$

$$\Rightarrow f^{(k)}(0) = (-1)^{k-1}(k-1)!, \quad k \geqslant 0.$$

We show that f is real-analytic. To this end, we write $f(x) = T_m(x) + R_m(x)$ with

$$T_m(x) = \sum_{k=0}^{m} \frac{1}{k!} f^{(k)}(0)\, x^k = \sum_{k=1}^{m} \frac{(-1)^{k-1}}{k}\, x^k.$$

<u>$0 \leq x \leq 1$</u>: From Thm 42 with $p = m+1$ we obtain

$$|R_m(x)| = \left| \frac{1}{(m+1)!} f^{(m+1)}(\vartheta x)\, x^{m+1} \right| = \frac{1}{m+1} \left(\frac{x}{1+\vartheta x} \right)^{m+1} \to 0, \quad m \to \infty.$$

<u>$-1 < x \leq 0$</u>: From Thm 42 with $p = 1$ we obtain

$$|R_m(x)| = \left| \frac{1}{m!} f^{(m+1)}(\vartheta x)(1-\vartheta)^m x^{m+1} \right|$$

$$= \underbrace{\left(\frac{1-\vartheta}{1+\vartheta x} \right)}_{\in (0,1)}^{m} \frac{|x|^{m+1}}{1+\vartheta x} \to 0, \quad m \to \infty.$$

$\Rightarrow$ For $x \in (-1, 1]$ we have

$$\log(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k \quad \text{"Mercator series"}$$

In particular: $\log 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}$.

iii) Let $f: (-1, \infty) \to \mathbb{R}$, $f(x) = (1+x)^\alpha$, $\alpha \in \mathbb{R}$ fixed.

Then, by a similar reasoning as in ii), we can show that for $x \in (-1, 1)$

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k, \quad \text{"Binomial series"}$$

where $\binom{\alpha}{k} := \frac{1}{k!} f^{(k)}(0) = \frac{1}{k!} \alpha(\alpha-1)\cdots(\alpha-k+1)$.

Chapter 5. Integration

Integration evolved over centuries from geometric and physical considerations related to the determination of areas and volumes.

<u>Definition</u>. Let $[a,b]$, $a<b$, be a compact interval. By a <u>partition</u> P of $[a,b]$ we mean a set of points $x_0, x_1, \dots, x_n$ with $a = x_0 < x_1 < \dots < x_n = b$. Suppose $f: [a,b] \to \mathbb{R}$ is a bounded function. We define

$$M_i := \sup_{x \in [x_{i-1}, x_i]} f(x), \quad m_i := \inf_{x \in [x_{i-1}, i]} f(x), \quad i = 1, \dots, n,$$

$$\mathcal{U}(P,f) := \sum_{i=1}^{n} M_i (x_i - x_{i-1}), \quad L(P,f) := \sum_{i=1}^{n} m_i (x_i - x_{i-1}),$$

"upper sum" "lower sum"

and

$$\int_a^{b\,*} f(x)\,dx := \inf\{\mathcal{U}(P,f) : P \text{ is partition of } [a,b]\},$$

"upper Riemann integral"

$$\int_{a\,*}^{b} f(x)\,dx := \sup\{L(P,f) : P \text{ is partition of } [a,b]\}.$$

"lower Riemann integral"

If $\int_a^{b*} f(x)\,dx = \int_{a*}^{b} f(x)\,dx$, then we say that the function f is <u>Riemann-integrable</u> on $[a,b]$ and the <u>Riemann integral</u> of f on $[a,b]$ is defined as

$$\int_a^b f(x)\,dx := \int_a^{b*} f(x)\,dx.$$

Moreover, we define the set

$$\mathcal{R}([a,b]) := \{ f : [a,b] \to \mathbb{R} : f \text{ is Riemann-integrable} \}$$

and introduce the conventions

$$\int_a^a f(x)\,dx := 0, \quad \int_b^a f(x)\,dx = -\int_a^b f(x)\,dx \quad (a<b).$$

<u>Remarks.</u> i) For every bounded function $f:[a,b]\to\mathbb{R}$ we have

$$\int_a^{b*} f(x)\,dx,\ \int_{a*}^{b} f(x)\,dx \in \mathbb{R}$$

and $\int_{a*}^{b} f(x)\,dx \leq \int_a^{b*} f(x)\,dx$.

ii) Not every bounded function is Riemann-integrable, e.g., consider $f:[0,1]\to\mathbb{R}$,

$$f(x) := \begin{cases} 1, & x \in \mathbb{Q}\cap[0,1] \\ 0, & \text{else} \end{cases},$$ then we have

$$\int_0^{1*} f(x)\,dx = 1 \quad \text{but} \quad \int_{0*}^{1} f(x)\,dx = 0.$$

iii) It is not important which letter we choose to represent the variable of integration:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(\alpha)\,d\alpha = \cdots$$

Example. Let $f:[a,b]\to\mathbb{R}$, $f(x)=\begin{cases} y_1, & x=a \\ c, & a<x<b \\ y_2, & x=b \end{cases}$,

where y_1, y_2 and c are fixed real constants.
For any partition P of $[a,b]$ we have

$$U(P,f) = \max\{y_1,c\}(x_1-a) + \sum_{i=2}^{n-1} c(x_i-x_{i-1}) + \max\{y_2,c\}(b-x_{n-1})$$

$$= \max\{y_1,c\}(x_1-a) + c(x_{n-1}-x_1) + \max\{y_2,c\}(b-x_{n-1})$$

As x_1 can be chosen arbitrarily close to a, and x_{n-1} can be chosen arbitrarily close to b, we obtain $\int_a^{b*} f(x)\,dx \leq c(b-a)$. Similarly, we obtain $\int_{*a}^{b} f(x)\,dx \geq c(b-a)$, which shows that $f\in\mathcal{R}([a,b])$ with $\int_a^b f(x)\,dx = c(b-a)$.

Definition. Let P and P^* be partitions of $[a,b]$. We say that P^* is a refinement of P if every point of P is a point of P^*.
Given two partitions P_1 and P_2 of $[a,b]$, we call $P^* = P_1\cup P_2$ their common refinement.

<u>Theorem 44.</u> Let $f:[a,b]\to\mathbb{R}$ be bounded, and let P, P^* be partitions of $[a,b]$ such that P^* is a refinement of P. Then

$$L(P,f) \le L(P^*,f) \text{ and } \mathcal{U}(P^*,f) \le \mathcal{U}(P,f).$$

<u>Proof.</u> We only show the first inequality. To this end, it is sufficient to consider the case that P^* contains just one point more than P. Let this point be x^* and suppose that $x^* \in (x_{j-1}, x_j)$, where $a = x_0 < x_1 < \dots < x_n = b$ are the points of P. Then we have

$$m_j = \inf_{x\in[x_{j-1},x_j]} f(x) \le \min\left\{ \inf_{x\in[x_{j-1},x^*]} f(x), \inf_{x\in[x^*,x_j]} f(x) \right\},$$

and thus $L(P,f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$

$$= \sum_{\substack{i=1\\ i\neq j}}^{n} m_i(x_i-x_{i-1}) + \underbrace{m_j(x_j - x^*)}_{\le \inf_{x\in[x^*,x_j]} f(x)\,(x_j-x^*)} + \underbrace{m_j(x^*-x_{j-1})}_{\le \inf_{x\in[x_{j-1},x^*]} f(x)\,(x^*-x_{j-1})}$$

$$\le L(P^*,f).$$ □

<u>Theorem 45</u> Let $f:[a,b]\to\mathbb{R}$ be bounded. Then we have $f\in\mathcal{R}([a,b])$ if and only if for every $\varepsilon>0$ there exists a partition P of $[a,b]$ such that $\mathcal{U}(P,f) - L(P,f) < \varepsilon$.