the function $f \cdot g$ is differentiable at ξ with

$$(f \cdot g)'(\xi) = f'(\xi) g(\xi) + f(\xi) g'(\xi) \quad \text{"product rule"},$$

and, if $g(\xi) \neq 0$, the function $\frac{f}{g}$ is differentiable at ξ with

$$\left(\frac{f}{g}\right)'(\xi) = \frac{f'(\xi) g(\xi) - f(\xi) g'(\xi)}{g(\xi)^2} \quad \text{"quotient rule"}.$$

ii) Let $f : I \to J$ and $g : J \to \mathbb{R}$ two functions such that f is differentiable at $\xi \in I$ and g is differentiable at $\eta = f(\xi)$. Then $g \circ f : I \to \mathbb{R}$ is differentiable at ξ with

$$(g \circ f)'(\xi) = g'(f(\xi)) f'(\xi) \quad \text{"chain rule"}.$$

<u>Proof.</u> i) For any $x \in I \setminus \{\xi\}$ we have

$$\frac{\alpha f(x) + \beta g(x) - \alpha f(\xi) - \beta g(\xi)}{x - \xi}$$

$$= \alpha \frac{f(x) - f(\xi)}{x - \xi} + \beta \frac{g(x) - g(\xi)}{x - \xi} \xrightarrow[x \to \xi]{} \alpha f'(\xi) + \beta g'(\xi),$$

which proves the sum rule. Moreover, for $x \in I \setminus \{\xi\}$, we have

$$\frac{f(x)g(x) - f(\xi)g(\xi)}{x-\xi} = \frac{f(x)-f(\xi)}{x-\xi}\, g(x) + f(\xi)\,\frac{g(x)-g(\xi)}{x-\xi}$$
$$\xrightarrow[x\to\xi]{} f'(\xi)g(\xi) + f(\xi)g'(\xi),$$

proving the product rule. Furthermore, for $x \in U_\delta(\xi)\setminus\{\xi\}$, where $\delta > 0$ is chosen small enough to ensure that $g(x) \neq 0$ on $U_\delta(\xi)$, we have

$$\frac{\frac{f(x)}{g(x)} - \frac{f(\xi)}{g(\xi)}}{x-\xi} = \frac{1}{g(x)g(\xi)}\,\frac{f(x)g(\xi) - f(\xi)g(x)}{x-\xi}$$

$$= \frac{1}{g(x)g(\xi)}\left\{\frac{f(x)-f(\xi)}{x-\xi}\, g(\xi) - \frac{g(x)-g(\xi)}{x-\xi}\, f(\xi)\right\}$$

$$\xrightarrow[x\to\xi]{} \frac{1}{g(\xi)^2}\left\{f'(\xi)g(\xi) - f(\xi)g'(\xi)\right\},$$

which proves the quotient rule.

ii) We define $r : J \to \mathbb{R}$ by

$$r(y) := \begin{cases} \frac{g(y)-g(\eta)}{y-\eta} - g'(\eta) & , y \in J\setminus\{\eta\} \\ 0 & , y = \eta. \end{cases}$$

$\Rightarrow$ r is continuous at $y=\eta$ with $r(\eta) = 0$.

Using this function r, we can write for $y \in J$

$$g(y) - g(\eta) = g'(\eta)(y-\eta) + r(y)(y-\eta)$$

$\Rightarrow (y = f(x))$ For $x \in I$ we have

$$g(f(x)) - g(f(\xi)) = \big(g'(f(\xi)) + r(f(x))\big)\big(f(x) - f(\xi)\big)$$

$\Rightarrow$ For $x \in I \setminus \{\xi\}$ we have

$$\frac{g(f(x)) - g(f(\xi))}{x - \xi} = \big(g'(f(\xi)) + r(f(x))\big)\,\frac{f(x) - f(\xi)}{x - \xi}$$

$$\xrightarrow[x \to \xi]{} g'(f(\xi))\, f'(\xi). \qquad \square$$

<u>Examples.</u> i) Let $p: \mathbb{R} \to \mathbb{R}$, $p(x) = \sum_{k=0}^{m} a_k x^k$, be a polynomial, then, by the sum rule, p is differentiable on $\mathbb{R}$ with derivative

$$p'(x) = \frac{d}{dx} \sum_{k=0}^{m} a_k x^k = \sum_{k=0}^{m} a_k \frac{d}{dx} x^k = \sum_{k=1}^{m} a_k k x^{k-1}.$$

ii) Let $r: I \to \mathbb{R}$, $r(x) = \frac{p(x)}{q(x)}$, be a rational function, where $q(x) \neq 0$ on I. Then, by the quotient rule, r is differentiable on I with

$$r'(x) = \frac{p'(x)q(x) - p(x)q'(x)}{q(x)^2}.$$

In particular, for $k \in \mathbb{N}$ and $x \neq 0$, we have

$$\frac{d}{dx}\frac{1}{x^k} = -k\,x^{-k-1}.$$

iii) Let $a > 0$ and $f: \mathbb{R} \to (0, \infty)$, $f(x) = a^x = e^{x \log a}$.

Then, by the chain rule, f is differentiable on $\mathbb{R}$ with

$$f'(x) = e^{x \log a} \log a = a^x \log a .$$

iv) Using the series representation and the Cauchy product, we can show that the following addition formulas hold for sine and cosine:

$$\sin(x+y) = \sin x \cos y + \sin y \cos x$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y , \quad x, y \in \mathbb{R}.$$

We can use these formulas to differentiate sine and cosine, e.g., we have for any $\xi \in \mathbb{R}$

$$\lim_{x \to \xi} \frac{\sin x - \sin \xi}{x - \xi} = \lim_{x \to 0} \frac{\sin(x+\xi) - \sin \xi}{x}$$

$$= \lim_{x \to 0} \frac{\sin x \cos \xi + \sin \xi \cos x - \sin \xi}{x}$$

$$= \lim_{x \to 0} \cos \xi \frac{\sin x}{x} + \lim_{x \to 0} \sin \xi \frac{\cos x - 1}{x} = \cos \xi .$$

$\Rightarrow$ The sine is differentiable on $\mathbb{R}$ with $\frac{d}{dx} \sin x = \cos x$.

Similarly, we can show that cosine is differentiable on $\mathbb{R}$ with $\frac{d}{dx} \cos x = - \sin x$.

Moreover, using the quotient rule we see that

$$\tan x := \frac{\sin x}{\cos x} , \text{ for } x \neq \frac{\pi}{2} + k\pi , k \in \mathbb{Z},$$

is differentiable with

$$\frac{d}{dx} \tan x = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} .$$

Next, we deal with the differentiation of inverse functions.

<u>Theorem 36.</u> Let $f: I \to \mathbb{R}$ be strictly monotonic and continuous on I, and suppose that f is differentiable at $\xi \in I$ with $f'(\xi) \neq 0$. Then the inverse $f^{-1}: f(I) \to I$ is differentiable at $\eta = f(\xi)$ with
$$(f^{-1})'(\eta) = \frac{1}{f'(f^{-1}(\eta))} = \frac{1}{f'(\xi)}.$$

<u>Proof.</u> The existence of f^{-1} and its continuity is clear from Thm 33. For any $y \in f(I) \setminus \{\eta\}$ we define $x := f^{-1}(y) \neq \xi$ (f is strictly monotonic).
$$\Rightarrow \frac{f^{-1}(y) - f^{-1}(\eta)}{y - \eta} = \frac{x - \xi}{f(x) - f(\xi)} = \left(\frac{f(x) - f(\xi)}{x - \xi}\right)^{-1} \to \frac{1}{f'(\xi)},$$
as $y \to \eta$, where we use that $x = f^{-1}(y) \to f^{-1}(\eta) = \xi$, as $y \to \eta$. $\square$

<u>Examples.</u> i) We know that $f = \exp$ satisfies the assumptions of Thm 36, hence $\log: (0,\infty) \to \mathbb{R}$ is differentiable on $(0,\infty)$ with
$$\frac{d}{dx} \log x = \frac{1}{\exp(\log x)} = \frac{1}{x}, \quad x > 0.$$

ii) Let $\alpha \in \mathbb{R}$ and $f: (0,\infty) \to \mathbb{R}$, $f(x) = x^\alpha = e^{\alpha \log x}$.
Then, by the chain rule, f is differentiable on $(0,\infty)$ with

$$\frac{d}{dx} x^\alpha = \frac{d}{dx} e^{\alpha \log x} = e^{\alpha \log x} \cdot \frac{\alpha}{x} = \alpha x^{\alpha - 1}.$$

Next, we deal with local extrema and mean value theorems.

Definition. The function $f: D \to \mathbb{R}$ $(D \subset \mathbb{R})$ has a local maximum at $\xi \in D$ if there exists some $\varepsilon > 0$ such that $f(x) \leq f(\xi)$ for all $x \in \mathcal{U}_\varepsilon(\xi) \cap D$.
The function f has a local minimum at $\xi \in D$ if there exists some $\varepsilon > 0$ such that $f(x) \geq f(\xi)$ for all $x \in \mathcal{U}_\varepsilon(\xi) \cap D$.
Local minima and local maxima are also called local extrema.

Theorem 37. Let $f: (a,b) \to \mathbb{R}$ have a local extremum at $\xi \in (a,b)$ and suppose that f is differentiable at ξ. Then $f'(\xi) = 0$.

Proof. As f is differentiable at ξ, we know

$$f'(\xi) = f'_+(\xi) = f'_-(\xi).$$

W.l.o.g. let us assume that f has a local maximum at ξ. Then we have

$$f'_+(\xi) = \lim_{x\to\xi+} \underbrace{\frac{f(x)-f(\xi)}{x-\xi}}_{\leq 0} \leq 0, \quad f'_-(\xi) = \lim_{x\to\xi-} \underbrace{\frac{f(x)-f(\xi)}{x-\xi}}_{\geq 0} \geq 0$$

$\Rightarrow f'(\xi) = 0.$ $\square$

<u>Remark</u>. The condition $f'(\xi)=0$ is only necessary but not sufficient for a local extremum. For example, consider $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^3$, then we have $f'(0)=0$ but f does not have a local extremum at 0.

<u>Theorem 38</u> (Rolle's Theorem)
Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Further, suppose that $f(a)=f(b)$, then there exists some $\xi\in(a,b)$ such that $f'(\xi)=0$.

<u>Proof.</u> By Thm 34 we can find $\xi_1,\xi_2\in[a,b]$ such that $f(\xi_1)=\min_{x\in[a,b]} f(x)$, $f(\xi_2)=\max_{x\in[a,b]} f(x)$.

If $\xi_1\in(a,b)$ or $\xi_2\in(a,b)$, then the statement follows from Thm 37. Let us assume now that $\xi_1,\xi_2\in\{a,b\}$.

$\Rightarrow f(\xi_1) = f(\xi_2) = f(a) = f(b)$

$\Rightarrow f$ is constant on $[a,b]$

$\Rightarrow f'(\xi) = 0$ for every $\xi \in (a,b)$. $\square$

<u>Theorem 39.</u> (Generalised Mean Value Theorem)

Let $f, g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then there exists some $\xi \in (a,b)$ such that

$$(f(b) - f(a))\, g'(\xi) = (g(b) - g(a))\, f'(\xi).$$

<u>Proof.</u> The proof follows immediately from the application of Thm 38 to the function

$$h : [a,b] \to \mathbb{R}, \quad h(x) = (f(b) - f(a))\, g(x) - (g(b) - g(a))\, f(x). \quad \square$$

<u>Remarks.</u> i) If $g'(x) \neq 0$ on (a,b), then $g(b) \neq g(a)$ (by Thm. 38), so we can write the statement of Thm. 39 as

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

ii) In the special case $g(x) = x$, we obtain

$$f(b) - f(a) = f'(\xi)(b-a) \quad \text{for some } \xi \in (a,b).$$

This statement often is called <u>Mean Value Theorem</u>.

iii) As a consequence, we can observe for any function $f:[a,b]\to\mathbb{R}$, which is continuous on $[a,b]$ and differentiable on (a,b), that f is constant on $[a,b] \iff f'(x)=0$ on (a,b).

<u>Theorem 40.</u> Let $f:(a,b)\to\mathbb{R}$ be differentiable. Then we have

i) $f'(x)>0$ on (a,b) $\Rightarrow$ f is strictly increasing on (a,b)

ii) $f'(x)<0$ on (a,b) $\Rightarrow$ f is strictly decreasing on (a,b)

iii) $f'(x)\geq 0$ on $(a,b) \iff f$ is increasing on (a,b)

iv) $f'(x)\leq 0$ on $(a,b) \iff f$ is decreasing on (a,b).

<u>Proof.</u> "$\Rightarrow$" in all four parts: For $x_1,x_2\in(a,b)$ with $x_1<x_2$, by Thm 39, we find a $\xi\in(x_1,x_2)$ such that $f(x_2)-f(x_1)=f'(\xi)(x_2-x_1)$.

"$\Leftarrow$" in parts iii, iv: W.l.o.g. we assume that f is increasing. For any $\xi\in(a,b)$ we have

$$f'(\xi)=\lim_{x\to\xi}\underbrace{\frac{f(x)-f(\xi)}{x-\xi}}_{\geq 0}\geq 0. \qquad \square$$

<u>Example:</u> The tangent $\tan:(-\frac{\pi}{2},\frac{\pi}{2})\to\mathbb{R}$, $\tan x=\dfrac{\sin x}{\cos x}$, is strictly increasing on $(-\frac{\pi}{2},\frac{\pi}{2})$:

$$\frac{d}{dx} \tan x = \frac{1}{\cos^2 x} > 0.$$

<u>Theorem 41.</u> (L'Hôpital's Rule)

Let $f, g : (a,b) \to \mathbb{R}$ be differentiable functions with $g'(x) \neq 0$ on (a,b), where $-\infty \leq a < b \leq \infty$, and suppose that

i) $\lim_{x \to a+} f(x) = \lim_{x \to a+} g(x) = 0$ or

ii) $\lim_{x \to a+} f(x) = \lim_{x \to a+} g(x) = \infty$.

If $\lim_{x \to a+} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x \to a+} \frac{f(x)}{g(x)}$ exists and $\lim_{x \to a+} \frac{f(x)}{g(x)} = \lim_{x \to a+} \frac{f'(x)}{g'(x)}$.

<u>Proof.</u> We only focus on the case i) and $a > -\infty$ (the other cases are similar). We define $f(a) := g(a) := 0$ and choose some $b' \in (a,b)$.

$\Rightarrow$ f, g are continuous on $[a, b']$ and differentiable on (a, b'). By Thm 39, for any $x \in (a, b')$, we find some $\xi \in (a, x)$ such that $\frac{f(x)}{g(x)} = \frac{f(x) - f(0)}{g(x) - g(0)} = \frac{f'(\xi)}{g'(\xi)}$

$\Rightarrow$ As $x \to a+$, we obtain

$$\lim_{x\to a+}\frac{f(x)}{g(x)}=\lim_{\xi\to a+}\frac{f'(\xi)}{g'(\xi)}. \qquad \square$$

Examples i) We have for $a,b>0$

$$\lim_{x\to 0+}\frac{a^x-b^x}{x}=\lim_{x\to 0+}\frac{a^x\log a-b^x\log b}{1}$$

$$=\log a\lim_{x\to 0+}e^{(\log a)x}-\log b\lim_{x\to 0+}e^{(\log b)x}$$

$$=\log a-\log b=\log\frac{a}{b}.$$

ii) $$\lim_{x\to 0+}x\log x=-\lim_{x\to 0+}\frac{-\log x}{\frac{1}{x}}=-\lim_{x\to 0+}\frac{-\frac{1}{x}}{-\frac{1}{x^2}}$$

$$=-\lim_{x\to 0+}x=0.$$

Next, we deal with higher-order derivatives.

Definition. Suppose $f: I\to\mathbb{R}$ is differentiable on I. If $f': I\to\mathbb{R}$ is differentiable at $\xi\in I$, we say that f is twice differentiable at ξ and call

$f''(\xi):=(f')'(\xi)$ the second derivative of f

at ξ. If $f''(\xi)$ exists at every $\xi\in I$, we say that f is twice differentiable on I.

More generally, for some $n\in\mathbb{N}$, suppose that