Chapter 4. Differentiation

In this chapter, we study the rate of change of functions $f: I \to \mathbb{R}$, where $I \subset \mathbb{R}$ denotes an interval.

For a linear function $L: \mathbb{R} \to \mathbb{R}$, $L(x) = ax + b$, where $a, b \in \mathbb{R}$ are fixed, the rate of change from the point $\xi \in \mathbb{R}$ to the point $x \in \mathbb{R}$ is given by

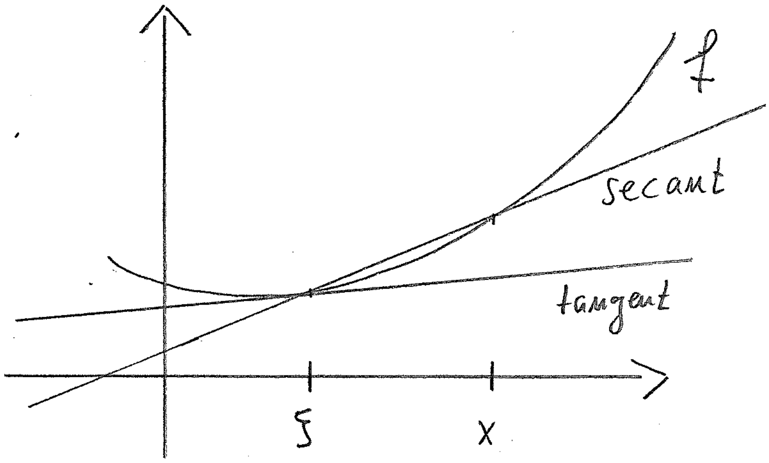
$$\frac{L(x) - L(\xi)}{x - \xi} = \frac{ax + b - a\xi - b}{x - \xi} = a.$$

Letting $x \to \xi$, we obtain the rate of change of L at the point ξ:

$$\lim_{x \to \xi} \frac{L(x) - L(\xi)}{x - \xi} = a.$$

For any function $f: I \to \mathbb{R}$, we can try to compute the rate of change at a point $\xi \in I$ by the same approach, i.e., we can try to find the limit

$$\lim_{x \to \xi} \frac{f(x) - f(\xi)}{x - \xi}.$$



Definition.

i) A function $f : I \to \mathbb{R}$ is differentiable at $\xi \in I$ if the limit

$$f'(\xi) := \lim_{x \to \xi} \frac{f(x) - f(\xi)}{x - \xi}$$

exists. In this case, we call $f'(\xi)$ the derivative of f at the point ξ. If f is differentiable at every point of I, we say that f is differentiable (on I), and the function $f' : I \to \mathbb{R}$ is the derivative of f.

ii) A function $f : I \to \mathbb{R}$ is $\left\{\begin{matrix} \text{right} \\ \text{left} \end{matrix}\right\}$ differentiable at $\xi \in I$ if the limit

$$f'_{\pm}(\xi) := \lim_{x \to \xi\pm} \frac{f(x) - f(\xi)}{x - \xi}$$

is well-defined and exists. In this case, we call $f'_{\pm}(\xi)$ the $\left\{\begin{matrix} \text{right} \\ \text{left} \end{matrix}\right\}$ derivative of f at ξ. If f is $\left\{\begin{matrix} \text{right} \\ \text{left} \end{matrix}\right\}$ differentiable at every point of I, then we say that f is $\left\{\begin{matrix} \text{right} \\ \text{left} \end{matrix}\right\}$ differentiable (on I), and the function $f'_{\pm} : I \to \mathbb{R}$ is the $\left\{\begin{matrix} \text{right} \\ \text{left} \end{matrix}\right\}$ derivative of f.

<u>Remark.</u> i) There are different ways to denote derivatives, e.g., we also write

$$f'(x) = \frac{df}{dx}(x) = \frac{d}{dx} f(x).$$

ii) A function $f:(a,b) \to \mathbb{R}$ is differentiable at $\xi \in (a,b)$ if and only if f is right and left differentiable at ξ and $f_+'(\xi) = f_-'(\xi)$. In this case, we have $f'(\xi) = f_+'(\xi) = f_-'(\xi)$.

iii) If $f: I \to \mathbb{R}$ is differentiable at $\xi \in I$, then f is continuous at ξ:

$$\lim_{x \to \xi} (f(x) - f(\xi)) = \lim_{x \to \xi} \left\{ \frac{f(x) - f(\xi)}{x - \xi} (x - \xi) \right\}$$

$$= \lim_{x \to \xi} \frac{f(x) - f(\xi)}{x - \xi} \cdot \lim_{x \to \xi} (x - \xi) = 0.$$

<u>Examples.</u> i) Let $k \in \mathbb{N}$ be a fixed integer and $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^k$. Then we have for $x \neq \xi$

$$\frac{f(x) - f(\xi)}{x - \xi} = \frac{x^k - \xi^k}{x - \xi} = \sum_{\nu=0}^{k-1} x^\nu \xi^{k-1-\nu}$$

$$\xrightarrow[x \to \xi]{} \sum_{\nu=0}^{k-1} \xi^\nu \xi^{k-1-\nu} = k \xi^{k-1}.$$

$\Rightarrow$ f is differentiable on $\mathbb{R}$ with $f'(x) = k x^{k-1}$.

ii) The exponential function $\exp: \mathbb{R} \to (0,\infty)$ is differentiable on $\mathbb{R}$ with $\frac{d}{dx}\exp(x) = \exp(x)$. To see this, let $\xi \in \mathbb{R}$ be a point in $\mathbb{R}$, then we have

$$\lim_{x\to\xi} \frac{\exp(x)-\exp(\xi)}{x-\xi} = \lim_{x\to\xi} \frac{e^x - e^\xi}{x-\xi}$$

$$= \lim_{x\to\xi} e^\xi \frac{e^{x-\xi}-1}{x-\xi} = e^\xi \lim_{x\to\xi} \frac{e^{x-\xi}-1}{x-\xi}$$

$$= e^\xi \lim_{t\to 0} \frac{e^t-1}{t} = e^\xi = \exp(\xi).$$

iii) The function $f: \mathbb{R}\to\mathbb{R}$, $f(x) = |x|$, is not differentiable at $\xi = 0$ as

$$f'_-(0) = \lim_{x\to 0-} \frac{|x|}{x} = -1 \quad \text{but}$$

$$f'_+(0) = \lim_{x\to 0+} \frac{|x|}{x} = 1.$$

<u>Theorem 35.</u>

i) Let $f, g: I \to \mathbb{R}$ be differentiable at $\xi \in I$. Then, for any $\alpha, \beta \in \mathbb{R}$, the function $\alpha f + \beta g$ is differentiable at ξ with

$$(\alpha f + \beta g)'(\xi) = \alpha f'(\xi) + \beta g'(\xi) \quad \text{"sum rule"},$$